College of the Holy Cross, Fall 2007
Math 131, Midterm 3 – Solutions
November 28

I. For each of the following functions find the derivative and simplify.

A. (5) $f(x) = \sin(2x)\cos(4x)$

Solution: By the product and chain rules:

$$f'(x) = -4\sin(2x)\sin(4x) + 2\cos(2x)\cos(4x).$$

B. (5) $g(x) = \dfrac{1 + \ln x}{1 - x}$

Solution: By the quotient rule:

$$g'(x) = \frac{(1-x)\frac{1}{x} - (1 + \ln(x))(-1)}{(1-x)^2} = \frac{1 - x + x + x\ln(x)}{x(1-x)^2} = \frac{1 + x\ln(x)}{x(1-x)^2}.$$

C. (5) $h(x) = \tan^{-1}(4x^2 + x)$

Solution: Using $\frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2}\frac{du}{dx}$,

$$h'(x) = \frac{8x + 1}{1 + (4x^2 + x)^2}.$$

D. (5) $k(x) = x^{\tan x}$

Solution: For this one we need to use logarithmic differentiation $\ln(k(x)) = \tan(x)\ln(x)$, so differentiating both sides with respect to x,

$$\frac{k'(x)}{k(x)} = \frac{\tan(x)}{x} + \sec^2(x)\ln(x).$$

Then

$$k'(x) = x^{\tan(x)}\left(\frac{\tan(x)}{x} + \sec^2(x)\ln(x)\right).$$

E. (5) Find the equation of the tangent line to the curve $x^2y^3 + 2y = 3x$ at the point $(2, 1)$.

Using implicit differentiation, $3x^2y^2y' + 2xy^3 + 2y' = 3$, so

$$y' = \frac{3 - 2xy^3}{3x^2y^2 + 2}.$$

The slope of the tangent is the value of y' when $x = 2$ and $y = 1$, which is $m = \frac{-1}{14}$. Then the point-slope form gives

$$y - 1 = \frac{-1}{14}(x - 2).$$

II. (15) A rocket is launched vertically and is tracked by a ground station 3 miles from the launch pad. What is the vertical speed of the rocket when its height above the ground is 4 miles and its distance to the ground station is increasing at 3600 miles per hour?

Solution: The ground station, the rocket, and the launch pad form a right triangle at all times. Call h the height of the rocket and z the distance from the rocket to the ground station. Then by the Pythagorean Theorem, at all times,

$$h^2 + 9 = z^2$$

Taking time derivatives,

$$2h\frac{dh}{dt} = 2z\frac{dz}{dt}.$$

The 3600 is $\frac{dz}{dt}$ when $h = 4$. When $h = 4$, by the Pythagorean Theorem again, $z = \sqrt{16+9} = 5$. So $2 \cdot 4 \cdot \frac{dh}{dt} = 2 \cdot 5 \cdot 3600$, and

$$\frac{dh}{dt} = \frac{36000}{8} = 4500$$

(miles per hour).

III. All parts of this question refer to the function $f(x) = \dfrac{x}{(2x+1)^2}$.

A. (2) What is the domain of $f(x)$?

Solution: The domain is the set of all real numbers different from $-1/2$: $(-\infty, -1/2) \cup (-1/2, +\infty)$.

B. (5) Find all critical numbers and determine where $f(x)$ is increasing and decreasing.

Solution: By the quotient and chain rules,

$$f'(x) = \frac{(2x+1)^2 - 4x(2x+1)}{(2x+1)^4} = \frac{1-2x}{(2x+1)^3}.$$

There is one critical number, at $x = 1/2$. Note: $x = -1/2$ is not a critical number since it is not in the domain of f. But we do have to take it into account in determining the sign of $f'(x)$. We get that

- f' is positive and f is increasing on $(-1/2, 1/2)$
- f' is negative and f is decreasing on $(-\infty, -1/2), (1/2, +\infty)$

C. (2) Find all local maximum and minimum values of $f(x)$.

Solution: By the first derivative test, $x = 1/2$ is a local maximum and $f(1/2) = \frac{1}{8}$. There are no local minima.

D. (3) Determine the concavity of $f(x)$, and find any inflection points given that

$$f''(x) = \frac{8(x-1)}{(2x+1)^4}.$$

Solution: The given formula for $f''(x)$ gives positive values for all $x > 1$ and negative values for all $x < 1$. So the graph is concave up for $x > 1$ and concave down for $x < 1$ (except at $x = -1/2$ where f is undefined). Concave up on $(1, +\infty)$ and concave down on $(-\infty, -1/2) \cup (-1/2, 1)$. $x = 1$ is an inflection point since the concavity changes there.

E. (2) Find all asymptotes for the graph of $f(x)$. Hint: You may use the following facts if necessary:

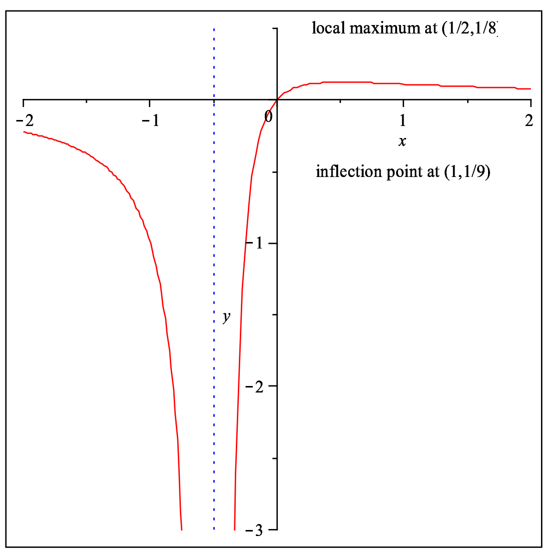
$$\lim_{x \to -0.5^+} f(x) = -\infty, \text{ and } \lim_{x \to -0.5^-} f(x) = -\infty.$$

Solution: The information in the Hint shows $x = -1/2$ is a vertical asymptote. There are no other vertical asymptotes since the domain of f contains all reals different from $-1/2$. There is also a horizontal asymptote for $f(x)$: Dividing top and bottom by x^2, we see

$$\lim_{x \to \pm\infty} \frac{x}{(2x+1)^2} = \lim_{x \to \pm\infty} \frac{1/x}{(2+1/x)^2} = 0$$

so $y = 0$ is the horizontal asymptote. (This could also be seen via L'Hopital's Rule.)

F. (6) Sketch the graph of $f(x)$ using the information above. Label any maximum and minimum values, inflection points, asymptotes, and any other points which help you to give a clear and accurate picture.



IV. All parts of this question refer to the function $f(x) = x^3 - 12x - 7$.

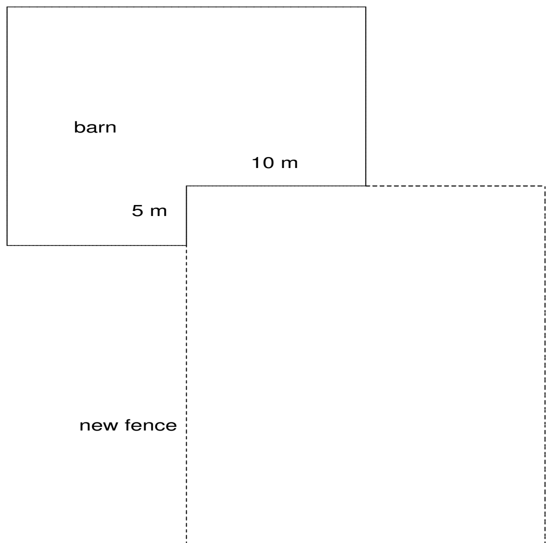
A. (5) Find the critical numbers of $f(x)$.

Solution: $f'(x) = 3x^2 - 12$, which is defined for all real x. So the critical numbers are the solutions of $3x^2 - 12 = 0$, or $x = \pm 2$.

B. (5) What does the Second Derivative Test tell you about the behavior of f at these critical numbers?

Solution: We have $f''(x) = 6x$. Since $f''(2) = 12 > 0$, $x = 2$ is a local *minimum*. Since $f''(-2) = -12 < 0$, $x = -2$ is a local *maximum*.

V. (15) A rectangular outdoor pen is to be added to a barn with a 5 meter by 10 meter corner notch as shown in the diagram below. If 85 meters of new fencing is available, what is the maximum area that can be enclosed? No fencing is needed along the walls of the barn. Be sure to say how you know your solution gives the maximum area.



Solution: Let x and y be the dimensions of the pen (taking x to be the total length of the side parallel to the 10m side of the notch and y to be the other side. The fencing used is $x + y + x - 10 + y - 5 = 2x + 2y - 15 = 85$, So $2x + 2y = 100$ and $y = 50 - x$. The area of the pen is $A(x) = xy = x(50 - x) = 50x - x^2$. The derivative is $A'(x) = 50 - 2x$ and there is one critical number $x = 25$. This is a maximum for the area function since there is only one critical number and $A'' = -2$ is negative. The final answer is: maximum area is $A(25) = 625$ square feet.

VI. Find the following limits. (Just an answer is not sufficient; you must show work for full credit.)

A. (3) $\lim_{x \to 0} \frac{x^3 + 2\sqrt{x}}{5x^3 + 2}$

Solution: Technically speaking, the correct answer to this one is that the limit is not defined since $\sqrt{x}$ is not defined for $x < 0$ and we need to include those x to take the limit. If we look only at $x \to 0^+$, then the answer is

$$\lim_{x \to 0^+} \frac{x^3 + 2\sqrt{x}}{5x^3 + 2} = 0$$

(This is not an indeterminate form, since the bottom is going to 2, not 0. So L'Hopital's Rule does not apply!)

Note: We goofed in designing this problem so either answer above will get full credit.

B. (4) $\lim_{x \to 2} \frac{e^{x^2} - e^4}{x - 2}$

Solution: This one is indeterminate of the form 0/0, so L'Hopital's rule does apply. We look at

$$\lim_{x \to 2} \frac{2xe^{x^2}}{1} = 4e^4.$$

L'Hopital's Rule says the original limit equals $4e^4$ as well.

C. (4) $\lim_{x \to 0} \frac{e^x - x - 1}{\cos x - 1}$

Solution: This is indeterminate of the form 0/0. Applying L'Hopital's rule, we look at

$$\lim_{x \to 0} \frac{e^x - 1}{-\sin x}$$

This is still indeterminate of the form 0/0, so we apply L'Hopital again:

$$\lim_{x \to 0} \frac{e^x}{-\cos x} = -1.$$

The original limit is also equal to -1.

D. (4) $\lim_{x \to \infty} \frac{\ln x}{\sqrt[4]{x}}$

Solution: This is indeterminate of the form ∞/∞ so L'Hopital's Rule applies. The quotient of the derivatives is

$$\frac{\frac{1}{x}}{\frac{1}{4x^{3/4}}} = \frac{4x^{3/4}}{x} = \frac{4}{x^{1/4}}$$

As $x \to \infty$, this goes to 0, so the original limit is equal to 0 also.