MATH 131 – Calculus for Physical and Life Sciences
Solutions for Exam 2
October 25, 2007

I. A. Consider the parametric curve $x = 4\cos(t)$, $y = 4\sin(t)$.

(4) 1. Give a Cartesian equation for this curve (eliminate the parameter t to obtain an equation in x and y) and describe the shape of the curve.

Solution: We have $\frac{x}{4} = \cos(t)$ and $\frac{y}{4} = \sin(t)$, so by the basic trigonometric identity,

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2(t) + \sin^2(t) = 1.$$

Hence $x^2 + y^2 = 16$. The parametric curve lies on the circle with center $(0, 0)$ and radius 4.

(4) 2. What portion of the curve is traced out for $0 \leq t \leq \frac{\pi}{4}$, and in which direction is the curve being traced? (Either explain in words or draw a picture.)

Solution: For $0 \leq t \leq \frac{\pi}{4}$, the parametric curve traces the arc of the circle starting at $(x, y) = (4, 0)$ and ending at $(x, y) = (2\sqrt{2}, 2\sqrt{2})$, in the counterclockwise direction.

B. Now consider the parametric curve $x = \cos(4t)$, $y = \sin(4t)$.

(3) 1. Give a Cartesian equation for this curve and describe the shape of the curve.

Solution: By the basic trigonometric identity,

$$x^2 + y^2 = \cos^2(4t) + \sin^2(4t) = 1.$$

Hence $x^2 + y^2 = 1$. The parametric curve lies on the circle with center $(0, 0)$ and radius 1.

(4) 2. What portion of the curve is traced out of $0 \leq t \leq \frac{\pi}{4}$, and in which direction is the curve being traced? (Either explain in words or draw a picture.)

Solution: For $0 \leq t \leq \frac{\pi}{4}$, this parametric curve traces the arc of the circle starting at $(x, y) = (1, 0)$ and ending at $(x, y) = (-1, 0)$ (that is, the top half of the circle), in the counterclockwise direction.

II. Apply the rules for differentiation to find the derivatives of the functions given below. Do NOT simplify your answers after completing the derivative (except to clean up your answers).

(4) A. $f(x) = x^3 + 5x^{\frac{2}{3}} + \sqrt{2}$

Solution: By the sum and product rules,

$$f'(x) = 3x^2 + \frac{10}{3}x^{\frac{-1}{3}}$$

(the $\sqrt{2}$ is constant, so the derivative of that term is zero).

(4) B. $g(x) = x^{\frac{5}{2}} - \frac{1}{x} + 7$

Solution: Rewriting $g(x)$ first, $g(x) = x^{\frac{5}{2}} - x^{-1} + 7$, so

$$g'(x) = \frac{5}{2}x^{\frac{3}{2}} + x^{-2}.$$

(4) C. $R(z) = (5z^2 - 2z^4)(z^3 - 1)$

Solution: Using the product rule,

$$\begin{aligned} R'(z) &= (5z^2 - 2z^4)\frac{d}{dz}(z^3 - 1) + (z^3 - 1)\frac{d}{dz}(5z^2 - 2z^4) \\ &= (5z^2 - 2z^4)(3z^2) + (z^3 - 1)(10z - 8z^3). \end{aligned}$$

The directions said not to simplify, so this is sufficient. (This derivative could also be computed by multiplying out $R(z)$ and applying the power rule repeatedly. If this was done, the result would be $R(z) = 5z^5 - 2z^7 - 5z^2 + 2z^4$, so $R'(z) = 25z^4 - 14z^6 - 10z + 8z^3$.)

(4) D. $s(t) = \dfrac{\sqrt{t} - 2e^t}{e^t + t}$

By the quotient rule:

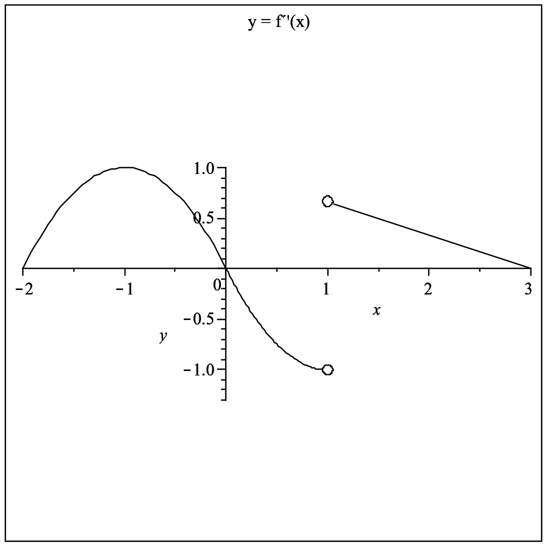
$$\begin{aligned} s'(t) &= \frac{(e^t + t)\frac{d}{dt}(\sqrt{t} - 2e^t) - (\sqrt{t} - 2e^t)\frac{d}{dt}(e^t + t)}{(e^t + t)^2} \\ &= \frac{(e^t + t)\left(\frac{1}{2}t^{-1/2} - 2e^t\right) - (\sqrt{t} - 2e^t)(e^t + 1)}{(e^t + t)^2} \end{aligned}$$

(again, no further simplification necessary as per the directions).

(4) E. $P(x) = e^x - x^e + e^\pi$

Solution: The e^x is an exponential function, the x^e is a power function, and the e^π is constant: $P'(x) = e^x - ex^{e-1}$.

III. The following graph shows $y = f'(x)$ for some function $f(x)$.



(3) A. Over what intervals is $f(x)$ increasing? Over what intervals is $f(x)$ decreasing?

Solution: $f(x)$ is increasing on intervals where $f'(x) > 0$, so on $(-2, 0)$ and $(1, 3)$. Similarly $f(x)$ is decreasing on intervals where $f'(x) < 0$, so on $(0, 1)$.

(3) B. Over what intervals is $f(x)$ concave up? Over what intervals is $f(x)$ concave down?

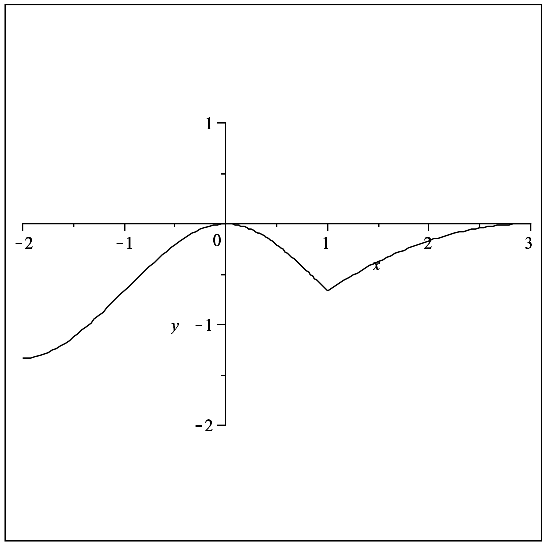
Solution: $f(x)$ is concave up on intervals where $f''(x) > 0$ or equivalently where $f'(x)$ is increasing. This is true here on $(-2, -1)$. Similarly $f(x)$ is concave down on intervals where $f''(x) < 0$, or equivalently where $f'(x)$ is decreasing. This is true here on $(-1, 1)$ and $(1, 3)$.

(3) C. What happens at $x = 1$ on the graph $y = f(x)$?

Solution: At $x = 1$, $f(x)$ is not differentiable because there is a "corner" point where the slope suddenly changes from -1 to approximately $2/3$.

(6) D. Sketch the graph $y = f(x)$ if it is given that $f(0) = 0$.

Solution: The actual graph of the function $f(x)$ is at the top of the next page.



Note on how this will be graded: The exact y-coordinates are not important here (except for the given $f(0) = 0$). We will be looking for the correct signs of the slopes, the concavities, and the "corner" at $x = 1$.

IV.

(4) A. State the limit definition of the derivative $f'(x)$.

Solution: The definition is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided that the limit exists.

(6) B. *Use the definition* to compute the derivative of $f(x) = x^2 + 2x + 1$. (Do not use the derivative rules.)

Solution: Applying the definition to $f(x) = x^2 + 2x + 1$,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) + 1 - (x^2 + 2x + 1)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h + 1 - (x^2 + 2x + 1)}{h} \text{ (multiply out)} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \text{ (cancel terms on top)} \\ &= \lim_{h \to 0} 2x + h + 2 \text{ (cancel } h\text{'s)} \\ &= 2x + 2. \end{aligned}$$

(2) C. Check your answer to part (b) by finding $f'(x)$ using the rules for differentiation. State each rule that you use.

Solution: By the sum, constant multiple, and power rules:

$$\frac{d}{dx}(x^2 + 2x + 1) = \frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) + \frac{d}{dx}(1) = 2x + 2$$

(the derivative of the constant 1 is zero).

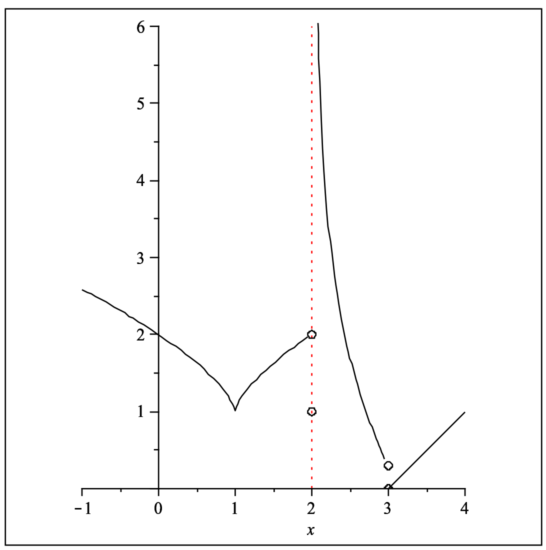
(4) D. Find the equation of the tangent line to the curve $y = x^2 + 2x + 1$ at $x = -3$.

Solution: The slope is $f'(3) = -6 + 2 = -4$ and the line goes through the point $(-3, f(-3)) = (-3, 4)$. The equation is $y - 4 = (-4)(x + 3)$, so $y = -4x - 8$.

(4) E. The position (in meters) of a moving object at time t (in seconds) is given by $y = f(t) = t^2 + 2t + 1$. What is the physical meaning of the statement $f'(2) = 6$ and what are the units of $f'(2)$?

Solution: The physical meaning is that the *velocity* of the object (the rate of change of position with respect to time) is 6 meters per second at time $t = 2$ seconds.

V. The following graph shows $y = f(x)$ for some function $f(x)$.



(5) A. At which x is $f(x)$ discontinuous? Justify your answer using the definition of continuity.

Solution: Recall that saying $f(x)$ is continuous at $x = a$ means that $\lim_{x \to a} f(x) = f(a)$. In other words, in order for $f(x)$ to be continuous at $x = a$, $f(a)$ must exist, the limit $\lim_{x \to a} f(x)$ must exist, and the limit must equal $f(a)$. This fails for the given function at two points. At $x = 2$, we have $f(2) = 1$, but $\lim_{x \to 2} f(x)$ does not exist because

$\lim_{x\to 2^-} f(x) = 2$, while $\lim_{x\to 2^+} f(x) = +\infty$. At $x = 3$, we have a similar but slightly different problem, $\lim_{x\to 3^+} f(x) = f(3) = 0$, but $\lim_{x\to 3^-} f(x) \neq 0$ (it's approximately 1/3 from the graph). Hence again $\lim_{x\to 3} f(x)$ does not exist, and $f(x)$ is not continuous at $x = 3$.

(5) B. At which x is $f(x)$ continuous but not differentiable? Explain.

Solution: $f(x)$ is continuous but not differentiable at the "cusp" point at $x = 1$. The derivative $f'(x)$ does not exist there because there is not a well-defined tangent line there. (Another possible explanation: the slopes of the tangents are positive on one side of the cusp and negative on the other side.)

VI. Compute the following limits. Show all work.

(5) A. $\lim_{x\to\pi} \sin\left(\frac{x}{2} - \pi\right)$

Solution: $\sin(x)$ is a continuous function for all x, so

$$\lim_{x\to\pi} \sin\left(\frac{x}{2} - \pi\right) = \sin\left(\lim_{x\to\pi} \frac{x}{2} - \pi\right) = \sin\left(\frac{-\pi}{2}\right) = -1.$$

(5) B. $\lim_{x\to 10} \frac{\sqrt{x-6} - 2}{x - 10}$

Solution: For this one, we will do "preliminary algebra" that is similar to what we use in computing the derivative of $\sqrt{x}$ by the limit definition – we multiply the top and bottom by the "conjugate radical" expression $\sqrt{x-6} + 2$ to start:

$$\begin{aligned}
\lim_{x\to 10} \frac{\sqrt{x-6} - 2}{x - 10} &= \lim_{x\to 10} \frac{\sqrt{x-6} - 2}{x - 10} \cdot \frac{\sqrt{x-6} + 2}{\sqrt{x-6} + 2} \\
&= \lim_{x\to 10} \frac{(x-6) - 4}{(x-10)\sqrt{x-6} + 2} \\
&= \lim_{x\to 10} \frac{x - 10}{(x-10)\sqrt{x-6} + 2} \\
&= \lim_{x\to 10} \frac{1}{\sqrt{x-6} + 2} \\
&= \frac{1}{\sqrt{4} + 2} \\
&= \frac{1}{4}.
\end{aligned}$$

(5) C. $\lim_{x\to 1^-} \frac{|x - 1|}{x^2 - 3x + 2}$

Solution: For the limit as $x \to 1^-$, we are interested only in x with $x < 1$. For $x < 1$, $|x-1| = -(x-1)$, and after factoring the bottom and cancelling, we get:

$$\begin{aligned}
\lim_{x\to 1^-} \frac{|x-1|}{x^2-3x+2} &= \lim_{x\to 1^-} \frac{-(x-1)}{(x-1)(x-2)} \\
&= \lim_{x\to 1^{-1}} \frac{-1}{x-2} \\
&= \frac{-1}{-1} = 1.
\end{aligned}$$

(5) D. Find all vertical and horizontal asymptotes to the graph $y = \dfrac{x^2+x-2}{x^2+5x+6}$.

Solution: The vertical asymptotes can be determined by factoring the top and bottom:

$$\frac{x^2+x-2}{x^2+5x+6} = \frac{(x+2)(x-1)}{(x+2)(x+3)}.$$

For $x \neq -2, -3$, we see

$$\frac{x^2+x-2}{x^2+5x+6} = \frac{x-1}{x+3}.$$

This does have a finite limit as $x \to -2$ (the limit equals -3). That is, there is a *removable discontinuity* of the original function at $x = -2$ (not a vertical asymptote). On the other hand, $\displaystyle\lim_{x\to -3^-} \frac{x-1}{x+3} = +\infty$, while $\displaystyle\lim_{x\to -3^+} \frac{x-1}{x+3} = -\infty$. That leaves $x = -3$ as the only vertical asymptote. The horizontal asymptote is $y = 1$, since

$$\lim_{x\to\pm\infty} \frac{x^2+x-2}{x^2+5x+6} = \lim_{x\to\pm\infty} \frac{1+\frac{1}{x}-\frac{2}{x^2}}{1+\frac{5}{x}+\frac{6}{x^2}} = 1.$$