College of the Holy Cross, Fall Semester, 2007
MATH 131, Section 01, Final Exam Solutions
December 10

1. [5 points each] Circle the number of the graph showing each of the following functions. Note that there is one more graph than there are formulas.

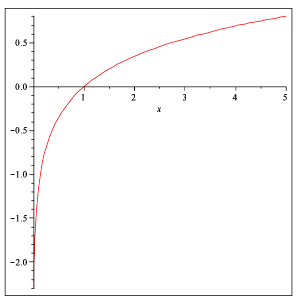
Solution:

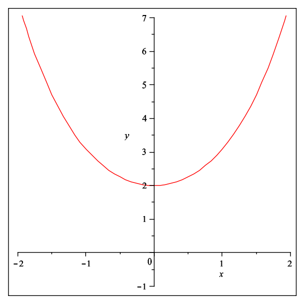
(a) $f(x) = 2 - 2\sin(\pi x)$ is graph III.

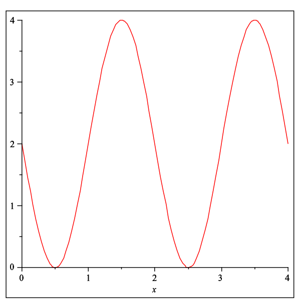
(b) $f(x) = \dfrac{x^2}{x^2 - 1}$ is graph V. (Note the vertical asymptotes at $x = \pm 1$.)

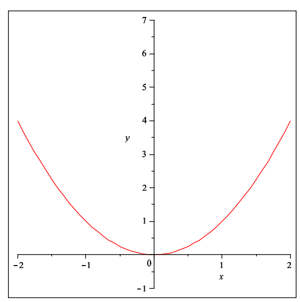
(c) $f(x) = \ln(\sqrt{x})$ is graph I.

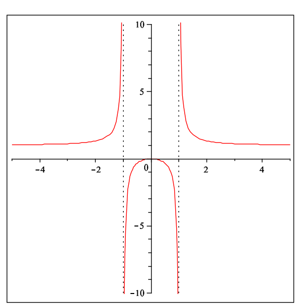
(d) $f(x) = e^x + e^{-x}$ is graph II. (This is not graph IV which has a somewhat similar shape, since $f(0) = e^0 + e^0 = 2$. This matches II but not IV.)

I:


II:


III:


IV:


V:


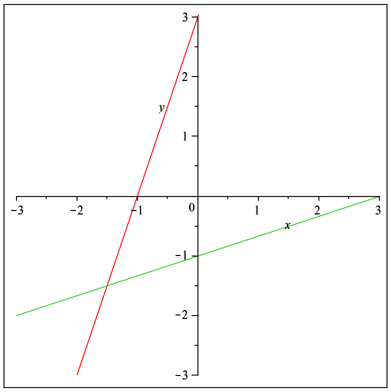
2.

(a) [5 points] What is the domain of the function $f(x) = \sqrt{x^2 - 4}$?

Solution: The domain is the set of all real x with $x^2 - 4 \geq 0$, or x in $(-\infty, 2] \cup [2, +\infty)$.

(b) [10 points] Find a formula for the inverse function of $f(x) = \frac{x}{3} - 1$ and plot $y = f(x)$ and $y = f^{-1}(x)$ on the same coordinate axes below.

Solution: To find the formula of f^{-1}, set $y = \frac{x}{3} - 1$ and solve for x: $x = 3(y + 1) = 3y + 3$. Switch variables to get $y = f^{-1}(x) = 3x + 3$. The graph $y = f(x)$ is a straight line with slope $\frac{1}{3}$ and y-intercept at -1. The graph $y = f^{-1}(x)$ is a straight line with slope 3 and y-intercept at $y = 3$:



(c) [5 points] Find the interval of all real x satisfying $-\pi/4 < \tan^{-1}(x) < \pi/3$.

Solution: Since the tangent function is increasing, this set of inequalities is the same as $\tan(-\pi/4) < x < \tan(\pi/3)$, or $-1 < x < \sqrt{3}$.

(d) [5 points] Express as a single logarithm: $\ln(6) - \ln(12) + 5\ln(2)$.

Solution: Using properties of logarithms, this equals $\ln\left(\frac{6 \cdot 2^5}{12}\right) = \ln(16)$.

3. Compute the following limits [5 points each]. Any legal method is OK.

(a) $\lim_{x \to 3^-} \frac{|x - 3|}{x^2 - 2x - 3}$

Solution: The denominator factors as $x^2-2x-3=(x-3)(x+1)$. But $\dfrac{|x-3|}{x-3}=-1$ when $x<3$ (which is what we want for the left-hand limit). Hence

$$\lim_{x\to 3^-}\frac{|x-3|}{x^2-2x-3}=\lim_{x\to 3^{-1}}\frac{-1}{x+1}=\frac{-1}{4}.$$

(b) $\displaystyle\lim_{x\to -1}\frac{x^3-x}{x^2+5x+4}$

Solution: This one is indeterminate of the form $0/0$ and can be done via "preliminary algebra" or L'Hopital's Rule. If we use L'Hopital,

$$\lim_{x\to -1}\frac{x^3-x}{x^2+5x+4}=\lim_{x\to -1}\frac{3x^2-1}{2x+5}=\frac{2}{3}.$$

(c) $\displaystyle\lim_{x\to +\infty}\frac{\ln(x^2)}{x^{3/2}}$

Solution: Indeterminate of the form ∞/∞, so we use L'Hopital, then simplify:

$$\lim_{x\to +\infty}\frac{\ln(x^2)}{x^{3/2}}=\lim_{x\to +\infty}\frac{\frac{2x}{x^2}}{(3/2)x^{1/2}}=\lim_{x\to +\infty}\frac{4}{3x^{3/2}}=0.$$

4.

(a) [5 points] State the limit definition of the derivative.

Solution: The derivative is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided that the limit exists.

(b) [10 points] Use the definition to compute $f'(x)$ for $f(x)=\dfrac{1}{x-1}$.

Solution: We compute

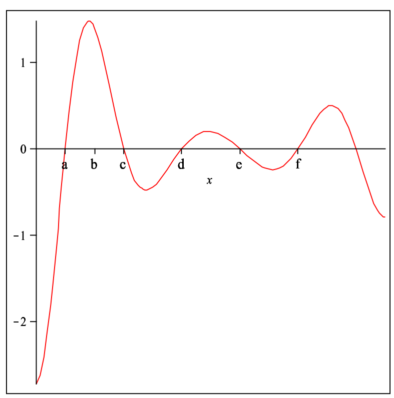
$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{x+h-1}-\frac{1}{x-1}}{h} \\
&= \lim_{h\to 0}\frac{(x-1)-(x+h-1)}{h(x-1)(x+h-1)} \\
&= \lim_{h\to 0}\frac{-h}{h(x-1)(x+h-1)} \\
&= \lim_{h\to 0}\frac{-1}{(x-1)(x+h-1)} \\
&= \frac{-1}{(x-1)^2}
\end{aligned}$$

(c) [10 points] Find the equation of the tangent line to the graph $y = \frac{1}{x-1}$ at the point $(5, 1/4)$.

Solution: The tangent line has slope $f'(5) = \frac{-1}{16}$ and passes through the point $(5, 1/4)$ on the graph of f. So the equation is

$$y - 1/4 = \frac{-1}{16}(x-5).$$

5. The following graph shows $y = g'(x)$, the derivative of some function $g(x)$.



Note: this is $y = g'(x)$, not $y = g(x)$.

(a) [5 points] At which of the points $x = 0, a, b, c, d, e, f$ does the function $g(x)$ have local maxima?

Solution: By the first derivative test, $g(x)$ has local maxima where $g'(x)$ changes from positive to negative: at $x = c, e$.

(b) [5 points] At which of the points $x = 0, a, b, c, d, e, f$ does the function $g(x)$ have local minima?

Solution: By the first derivative test, $g(x)$ has local minima where $g'(x)$ changes from negative to positive: at $x = a, d, f$.

(c) [5 points] On the interval (a, b), is $y = g(x)$ concave up or down?

Solution: On the interval (a, b), $g'(x)$ is increasing, so $(g')'(x) = g''(x) > 0$. This implies $y = g(x)$ is concave *up*.

6. Compute the following derivatives using the derivative rules. You need *not* simplify. [5 points each]

(a) $f(x) = 3x^5 + \frac{5}{x^3} + e^x$.

Solution: $f'(x) = 15x^4 - 15x^{-4} + e^x$.

(b) $g(x) = \dfrac{\sin(x) + 1}{\sin^{-1}(x) + 1}$.

Solution: By the quotient rule,

$$g'(x) = \frac{(\sin^{-1}(x) + 1)\cos(x) - (\sin(x) + 1)\frac{1}{\sqrt{1-x^2}}}{(\sin^{-1}(x) + 1)^2}.$$

(c) $h(x) = \sqrt{\ln(x) + 4\cos(\tan(x))}$.

Solution: By the chain rule and the derivative rules for $\ln, \cos, \tan$:

$$h'(x) = \frac{1}{2}\left(\ln(x) + 4\cos(\tan(x))\right)^{-1/2} \cdot \left(\frac{1}{x} - 4\sin(\tan(x))\sec^2(x)\right).$$

(d) $j(x) = (x^2 + 3)^{(x+2)}$.

Solution: Using logarithmic differentiation, $\ln(j(x)) = (x + 2)\ln(x^2 + 3)$. So then by the product rule

$$\frac{j'(x)}{j(x)} = \frac{2x(x + 2)}{x^2 + 2} + \ln(x^2 + 3)$$

Hence

$$j'(x) = (x^2 + 3)^{(x+2)}\left(\frac{2x(x + 2)}{x^2 + 2} + \ln(x^2 + 3)\right).$$

(e) Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 18xy$.

Solution: Differentiating implicitly,

$$3x^2 + 3y^2y' = 18y + 18xy'.$$

Then we solve algebraically for $y' = \frac{dy}{dx}$:

$$y' = \frac{18y - 3x^2}{3y^2 - 18x} = \frac{6y - x^2}{y^2 - 6x}.$$

7. All parts of this question refer to the function

$$f(x) = \frac{(x+1)^2}{x^2+1}.$$

(a) [5 points] Compute $f'(x)$ and simplify.

Solution: By the quotient and chain rules:

$$f'(x) = \frac{(x^2+1)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(x^2+1)^2} = \frac{-2(x^2-1)}{(x^2+1)^2}.$$

(b) [10 points] Find all critical numbers of f and determine the intervals on which f is increasing and decreasing.

Solution: $f'(x) = 0$ at $x = \pm 1$. These are the critical numbers. $f'(x) > 0$ so $f(x)$ is increasing on $(-1, 1)$. $f'(x) < 0$ so $f(x)$ is decreasing on $(-\infty, -1) \cup (1, +\infty)$.

(c) [5 points] **Given:** the second derivative in simplified form is

$$f''(x) = \frac{4x(x^2-3)}{(x^2+1)^3}.$$

Determine all points of inflection and the intervals where $y = f(x)$ is concave up and concave down.

Solution: From the form of the numerator, $f''(x) = 0$ at $x = 0, \pm\sqrt{3}$. $f''(x) > 0$ so $f(x)$ is concave up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, +\infty)$. $f''(x) < 0$ so $f(x)$ is concave down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$. The concavity changes at $-\sqrt{3}, 0, \sqrt{3}$, so all three are inflections.

(d) [5 points] Find the absolute maximum and minimum of $f(x)$ on the interval $[0, 5]$.

Solution: We compare $f(0) = 1$, $f(1) = 2$ (the value at the critical number in the interval), and $1 < f(5) = 18/13 < 2$. Hence the absolute minimum *on this interval* is $f(0) = 1$, and the absolute maximum is $f(1) = 2$.

8. [20 points] A potter forms 20 cm^3 of clay into a cylindrical tube. As she rolls the tube on a flat surface, its length L increases and its radius r decreases but the quantity of clay stays constant. If the length is increasing at the rate of .1 cm/sec, find the rate at which the radius is changing when the radius is 1cm. (*Note:* the volume of a cylinder of radius r and height h is $V = \pi r^2 h$.)

Solution: The volume of the clay is $20 = \pi r^2 L$ at all times. Taking derivatives with respect to time, and using the product rule,

$$0 = 2\pi r L \frac{dr}{dt} + \pi r^2 \frac{dL}{dt}.$$

When $r = 1$, $\frac{dL}{dt} = +0.1$ and from the volume equation $L = \frac{20}{\pi}$. Hence the radius is changing at the rate:

$$\frac{dr}{dt} = \frac{-\pi r^2 \frac{dL}{dt}}{2\pi r L} = \frac{-\pi (1)^2 (0.1)}{2\pi (1) \frac{20}{\pi}} = \frac{-\pi}{400} \doteq -0.0079 \text{ cm/sec.}$$

9. [20 points] A rectangular swimming pool with sides x, y is to be built with an area of 1800 square feet. The owner wants 5-foot wide decks along either side and 10-foot wide decks at the two ends. Find the dimensions of the rectangular lot of smallest area on which such a swimming pool can be built. Be sure you say how you know your answer gives a minimum of the area of the lot.

Solution: The area of the pool is $xy = 1800$, so $y = 1800/x$. If we let x be width of the pool, then the width of pool plus side decks is $x + 10$. The length of the pool plus the end decks is then $y + 20$. The total area of the rectangle containing the pool and the decks is

$$A = (x + 10)(y + 20) = xy + 10y + 20x + 200 = 2000 + \frac{18000}{x} + 20x.$$

Taking derivative with respect to x, and setting equal zero to find critical numbers, we find

$$0 = A'(x) = 20 - \frac{18000}{x^2}$$

so $x = \pm\sqrt{18000/20} = \pm\sqrt{900} = \pm 30$. The negative root is irrelevant here. So $x = 30$ is the only critical number we need to consider. By the equation $y = 1800/x$ we get $y = 60$. Now, at $x = 30$, $A''(x) = \frac{36000}{30^3} > 0$. Hence this gives a minimum of the area of the pool plus the decks. The final answer is that the lot should be $x + 10 = 40$ by $y + 20 = 80$ feet to fit the pool and the decks.

10. On and near the surface of the moon, the acceleration of gravity is -1.66 meters per second per second.

(a) [5 points] An astronaut throws a moon rock upward from a height of 1.5 meters above the surface with an initial velocity of 3 meters per second. Find the height of the rock above the surface as a function of time t in seconds.

Solution: We antidifferentiate once to find the velocity:

$$v(t) = -1.66t + c.$$

To get $v(0) = 3$, $c = 3$. Then we antidifferentiate again to get the height:

$$h(t) = -.83t^2 + 3t + d.$$

To get $h(0) = 1.5$, $d = 1.5$. So $h(t) = -.83t^2 + 3t + 1.5$.

(b) [3 points] How long does it take for the rock from part a to fall back to the surface of the moon?

Solution: We need to solve the equation $h(t) = -.83t^2 + 3t + 1.5 = 0$. The only way to do this is via the quadratic formula:

$$t = \frac{-3 \pm \sqrt{9 - 4(-.83)(1.5)}}{-1.66}.$$

The positive root is about $t = 4.06$ sec.

(c) [2 points] What is the velocity of the rock from part a when it strikes the surface of the moon?

Solution: Substitute $t = 4.06$ into the formula for $v(t)$ above: $v(4.06) \doteq -3.74$ m/sec.