Math 132: Calculus for Physical and Life Sciences 2
Problem Set 9
Due Friday, April 18, 2008, at the beginning of class.

General Directions: You must show all work for credit on these problems.

1. Determine whether each of the following series converges absolutely, converges conditionally, or diverges, using some combination of the Alternating Series Test, the Ratio Test, and the Integral Test.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^n}$

Solution: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{e^{n+1}}}{\frac{(-1)^n}{e^n}} \right| = \lim_{n\to\infty} \frac{1}{e} < 1.$$

Therefore, the series *converges absolutely.* Note: the fact that the ratio is a constant says this is actually a geometric series with ratio $\frac{-1}{e}$. Since this is less than 1 in absolute value, this series, and the series of absolute values both converge.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{4n-3}$

Solution: The terms in this series are not tending to zero: $\displaystyle\lim_{n\to\infty} \frac{2n+1}{4n-3} = \frac{1}{2} \neq 0.$ Therefore this series is *divergent.*

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n7^n}$

Solution: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)7^{n+1}}}{\frac{1}{n7^n}} \right| = \lim_{n\to\infty} \frac{n}{7(n+1)} = \frac{1}{7} < 1.$$

So this series is *absolutely convergent.* (Note that the series itself has all positive terms, so absolute convergence and "ordinary" convergence are the same in this example.)

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\ln(n)}$

Solution: The function $f(x) = \frac{1}{\ln(x)}$ has $f'(x) = \frac{-2}{x(\ln(x))^2} < 0$ for all $x > 1$ and $\displaystyle\lim_{x\to\infty} \frac{1}{\ln(x)} = 0$. Therefore, the series with the alternating signs converges by the Alternating Series Test. On the other hand, for all $x \geq 2$, $\ln(x) < x\ln(x)$, so $\frac{1}{\ln(x)} > \frac{1}{x\ln(x)}$ This implies that $\int_2^{\infty} \frac{1}{\ln(x)}\, dx$ diverges since $\int_2^{\infty} \frac{1}{x\ln(x)}\, dx$ diverges.

Therefore by the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges. The given series is *conditionally convergent.*

(e) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(Hint for (e): Try the Ratio Test on this one because of the powers and factorials. The limit you will need to compute is a 1^∞ indeterminate form. These are covered in Section 4.5 of the text if you need to review.)

Solution: Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

This limit is indeterminate of the form 1^∞. To evaluate it, we will apply L'Hopital's Rule after taking logarithms:

$$\ln\left(\left(1+\frac{1}{n}\right)^n\right) = n\ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}.$$

The limit here as $n \to \infty$ is indeterminate $0/0$. So by L'Hopital it should be the same as the limit

$$\lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Therefore

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 = e > 1,$$

and the series *diverges.*

2. For each of the following power series,

(i) Determine the radius of convergence R using the Ratio Test.

(ii) If $0 < R < \infty$, determine whether the series converges at $a \pm R$.

(iii) Based on (i) and (ii), give the interval of convergence.

(a) $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n}$

Solution: (i) Using the Ratio Test,

$$\left| \frac{\frac{(x-2)^{n+1}}{n+1}}{\frac{(x-2)^n}{n}} \right| = \frac{n}{n+1}|x-2|.$$

Therefore,

$$\lim_{n\to\infty} \frac{n}{n+1}|x-2| < 1 \Leftrightarrow |x-2| < 1,$$

or $1 < x < 3$. That is, the radius of convergence is $R = 1$.

(ii) If $x = 1$, the series is an alternating harmonic series, which converges by the Alternating Series Test. If $x = 3$, the series is a divergent harmonic series.

(iii) The interval of convergence is $[1, 3)$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

Solution: (i) Using the Ratio Test,

$$\left| \frac{\frac{2^{n+1}x^{n+1}}{(n+1)!}}{\frac{2^n x^n}{n!}} \right| = \frac{2|x|}{n+1}.$$

Therefore,

$$\lim_{n\to\infty} \frac{2|x|}{n+1} = 0 < 1$$

for all x. That is, the radius of convergence is $R = \infty$.

(ii) There are no endpoints to check for this series(!)

(iii) The interval of convergence is $(-\infty, \infty)$.

(c) $\displaystyle\sum_{n=0}^{\infty} n!(x-4)^n$

Solution: (i) Using the Ratio Test,

$$\left| \frac{(n+1)!(x-4)^{n+1}}{n!(x-4)^n} \right| = (n+1)|x-4|.$$

Therefore for any particular $x \neq 4$,

$$\lim_{n\to\infty} (n+1)|x-4| = +\infty.$$

If $x = 4$, then the series converges since all terms after the first are zero. That is, the radius of convergence is $R = 0$.

(ii) There are no endpoints to check for this series(!)

(iii) The interval of convergence consists of the single number $x = 4$.

3. Using the definition, find the Taylor series of each of the following functions $f(x)$ at the indicated a. Write the series using summation notation and also give the first four nonzero terms.

(a) $f(x) = e^{2x}$, $a = 0$.

Solution: Each time we differentiate e^{2x}, we get an extra factor of 2 by the Chain Rule. Therefore, $f^{(n)}(0) = 2^n$, and the Taylor Series is

$$\sum_{n=0} \frac{f^{(n)}(0)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{2^x x^n}{n!}.$$

The first four nonzero terms are:

$$1 + 2x + 2x^2 + \frac{4}{3}x^3.$$

(Note that this is the same series as in part (b) of question 2 above(!))

(b) $f(x) = \cos(x)$, $a = \pi/2$.
Solution: We have

$$\begin{aligned} f(\pi/2) &= 0 \\ f'(x) = -\sin(x) &\Rightarrow f'(\pi/2) = -1 \\ f''(x) = -\cos(x) &\Rightarrow f''(\pi/2) = 0 \\ f^{(3)}(x) = \sin(x) &\Rightarrow f^{(3)}(\pi/2) = 1 \\ f^{(4)}(x) = \cos(x) &\Rightarrow f^{(4)}(\pi/2) = 0, \end{aligned}$$

and thereafter the derivatives repeat according to the same pattern. The series has the form

$$\sum_{k=0}^{\infty} (-1)^{k+1} \frac{(x-\pi/2)^{2k+1}}{(2k+1)!},$$

with first four nonzero terms given by

$$-(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \frac{(x-\pi/2)^7}{7!}.$$

(Note, in this series, we can see the trig identity $\cos(x) = -\sin(x - \pi/2)$; do you see why?)

4. (a) Starting from the geometric series expansion for $f(x) = \dfrac{1}{1+x}$, determine a power series (with $a = 0$) representing the function $\ln(1+x)$.
Solution: By the geometric series formula with $r = -x$, we have

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^4 - \cdots,$$

and the series converges absolutely for all x satisfying $|x| < 1$. Therefore

$$\begin{aligned} \ln(1+x) &= \int \frac{dx}{x+1} \\ &= \int \sum_{n=0}^{\infty} (-x)^n \, dx = 1 - x + x^2 - x^3 + x^4 - \cdots \, dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots + C \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} + C, \end{aligned}$$

for some constant of integration C. The value of C can be determined by setting $x = 0$. Since $\ln(1) = 0$, $C = 0$, and we have

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

for all x with $|x| < 1$.

(b) What is the radius of convergence of your series from part (a)?

Solution:

$$\left| \frac{(-1)^n x^{n+1}/(n+1)}{(-1)^{n-1}x^n/n} \right| = \frac{n}{n+1}|x|.$$

Therefore

$$\lim_{n\to\infty} \frac{n}{n+1}|x| = |x| < 1$$

if and only if $|x| < 1$. That is, the radius of convergence is $R = 1$.

(c) Compute the Taylor series of $\ln(1+x)$ at $a = 0$. How do your answers to parts (a) and (c) compare?

Solution: We have for $f(x) = \ln(1+x)$,

$$\begin{aligned}
f(0) = \ln(1) &= 0 \\
f'(x) = \frac{1}{1+x} &\Rightarrow f'(0) = 1 \\
f''(x) = \frac{-1}{(1+x)^2} &\Rightarrow f''(0) = -1 \\
f^{(3)}(x) = \frac{2}{(1+x)^3} &\Rightarrow f^{(3)}(0) = 2 \\
&\vdots \\
f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} &\Rightarrow f^{(n)}(0) = (-1)^{n-1}(n-1)!.
\end{aligned}$$

Therefore, since the first term is zero anyway, we can start the summation from $n = 1$. Cancelling terms in the factorials, the Taylor series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}.$$

This is the same series as in part (a).

5. (a) Starting from the Taylor series for $f(t) = \sin(t)$, find a power series representation for the function

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\, dt.$$

Solution: The Taylor series for $\sin(t)$ is

$$\sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots .$$

Hence, dividing each term, then integrating, we get:

$$\begin{aligned}
\int_0^x \frac{\sin(t)}{t}\,dt &= \int_0^x 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots\, dt \\
&= x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \cdots \\
&= \sum_{k=0}^{\infty} (-1)^{2k+1} \frac{x^{2k+1}}{(2k+1)\cdot(2k+1)!}.
\end{aligned}$$

(b) Find a power series representation for the function

$$F(x) = \int_0^x \frac{1}{1-t^2}\,dt.$$

Solution: We expand in geometric series, and integrate:

$$\begin{aligned}
\int_0^x \frac{1}{1-t^2}\,dt &= \int_0^x \sum_{n=0}^{\infty} (t^2)^n\,dt \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}.
\end{aligned}$$

6. Compute the requested Taylor polynomials.

(a) The Taylor polynomial of degree $n = 3$ for $f(x) = \tan(x)$ at $a = 0$.
Solution: The derivatives and values we need:

$$\begin{aligned}
f(0) = \tan(0) &= 0 \\
f'(x) = \sec^2(x) &\Rightarrow f'(0) = 1 \\
f''(x) = 2\sec^2(x)\tan(x) &\Rightarrow f''(0) = 0 \\
f'''(x) = 2\sec^4(x) + 4\sec^2(x)\tan^2(x) &\Rightarrow f'''(0) = 2.
\end{aligned}$$

So the Taylor polynomial is

$$p_3(x) = x + \frac{x^3}{3}.$$

(b) The Taylor polynomial of degree $n = 6$ for $f(x) = \cos(4x)$ at $a = 0$.
Solution: We have

$$\begin{aligned}
f(0) &= 1 \\
f'(x) = -4\sin(4x) &\Rightarrow f'(0) = 0 \\
f''(x) = -16\cos(4x) &\Rightarrow f''(0) = -16 \\
f^{(3)}(x) = 64\sin(4x) &\Rightarrow f^{(3)}(0) = 0 \\
f^{(4)}(x) = 256\cos(4x) &\Rightarrow f^{(4)}(0) = 256 \\
f^{(5)}(x) = -1024\sin(4x) &\Rightarrow f^{(5)}(0) = 0 \\
f^{(4)}(x) = -4096\cos(4x) &\Rightarrow f^{(6)}(0) = 4096
\end{aligned}$$

So the Taylor polynomial is

$$p_6(x) = 1 - \frac{16}{2!}x^2 + \frac{256}{4!}x^4 - \frac{4096}{6!}x^6 = 1 - 8x^2 + \frac{32x^4}{3} - \frac{256x^6}{45}.$$

(c) The Taylor polynomial of degree $n = 4$ for $f(x) = \sqrt[3]{x-1}$ at $a = 2$.
Solution: We have

$$\begin{aligned} f(2) &= 1 \\ f'(x) = \frac{1}{3}(x-1)^{-2/3} &\Rightarrow f'(2) = \frac{1}{3} \\ f''(x) = \frac{-2}{9}(x-1)^{-5/3} &\Rightarrow f''(2) = \frac{-2}{9} \\ f^{(3)}(x) = \frac{10}{27}(x-1)^{-8/3} &\Rightarrow f^{(3)}(2) = \frac{10}{27} \\ f^{(4)}(x) = \frac{-80}{81}(x-1)^{-11/3} &\Rightarrow f^{(4)}(2) = \frac{-80}{81}. \end{aligned}$$

So the Taylor polynomial is

$$p_4(x) = 1 + \frac{1}{3}(x-2) - \frac{1}{9}(x-2)^2 + \frac{5}{81}(x-2)^3 - \frac{10}{243}(x-2)^4.$$

7. (a) Compute the Taylor polynomial of degree $n = 4$ for $f(x) = \sqrt{1+x}$ at $a = 0$.
Solution: We have

$$\begin{aligned} f(0) &= 1 \\ f'(x) = \frac{1}{2}(1+x)^{-1/2} &\Rightarrow f'(0) = \frac{1}{2} \\ f''(x) = \frac{-1}{4}(1+x)^{-3/2} &\Rightarrow f''(0) = \frac{-1}{4} \\ f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2} &\Rightarrow f^{(3)}(0) = \frac{3}{8} \\ f^{(4)}(x) = \frac{-15}{16}(1+x)^{-7/2} &\Rightarrow f^{(4)}(0) = \frac{-15}{16}. \end{aligned}$$

So the Taylor polynomial is

$$p_4(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4.$$

(b) Use your polynomial to approximate $\sqrt{1.1}$. (What x should you use to do this?)
Solution: Note that $\sqrt{1.1} = f(0.1) \doteq p_4(0.1)$. So we substitute $x = 0.1$ into $p_4(x)$ to get our approximation:

$$\sqrt{1.1} \doteq p_4(0.1) = 1 + \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 + \frac{1}{16}(0.1)^3 - \frac{5}{128}(0.1)^4 = 1.048808594.$$

(c) What is the error of your approximation, taking a calculator value for $\sqrt{1.1}$ as the exact value?

Solution: Taking $\sqrt{1.1} = 1.048808848$ as the exact value, the error in the approximation from (b) is

$$1.048808848 - 1.048808594 = 2.54 \times 10^{-7}.$$

(Note that the first 6 decimal places agree!)

8. Use Taylor series, not L'Hopital's Rule, to evaluate each of the following limits.

(a) $\lim_{x\to 0} \dfrac{1-\cos(x)}{x(e^x-1)}$.

Solution: Writing $\cdots$ for a sum of terms of higher degree in x, we have $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$ and $e^x = 1 + x + \frac{x^2}{2} + \cdots$, so ,

$$\frac{1-\cos(x)}{x(e^x-1)} = \frac{\frac{x^2}{2} - \frac{x^4}{24} + \cdots}{x^2 + \frac{x^3}{2} + \cdots} = \frac{\frac{1}{2} - \frac{x^2}{24} + \cdots}{1 + \frac{x}{2} + \cdots},$$

and

$$\lim_{x\to 0} \frac{1-\cos(x)}{x(e^x-1)} = \lim_{x\to 0} \frac{\frac{1}{2} - \frac{x^2}{24} \cdots}{1 + \frac{x}{2} + \cdots} = \frac{1}{2}.$$

(b) $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \tan^{-1}(x)}$.

Solution: Proceeding as in part (a),

$$\begin{aligned} e^x - e^{-x} - 2x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) - \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) - 2x \\ &= \frac{x^3}{3} + \frac{x^5}{60} + \cdots. \end{aligned}$$

By an example done in class,

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots,$$

so

$$x - \tan^{-1}(x) = \frac{x^3}{3} - \frac{x^5}{5} + \cdots.$$

Therefore,

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \tan^{-1}(x)} = \lim_{x\to 0} \frac{\frac{x^3}{3} + \frac{x^5}{60} + \cdots}{\frac{x^3}{3} - \frac{x^5}{5} + \cdots} = \lim_{x\to 0} \frac{\frac{1}{3} + \frac{x^2}{60} + \cdots}{\frac{1}{3} - \frac{x^2}{5} + \cdots} = 1.$$