Math 132: Calculus for Physical and Life Sciences 2
Solutions for Problem Set 7
Due Friday, April 4, 2008, at the beginning of class.

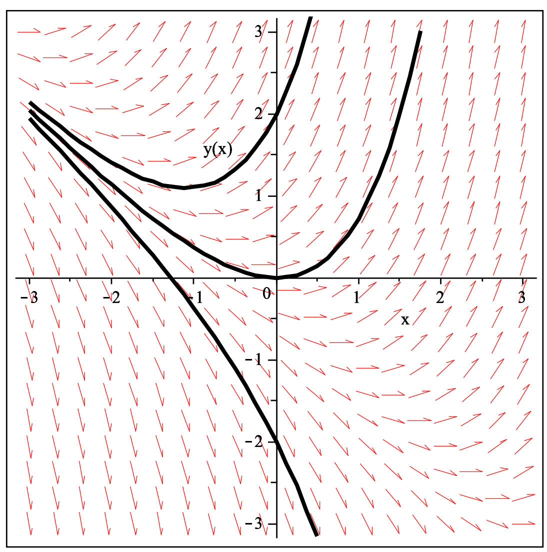
General Directions: You must show all work for credit on these problems.

1. All parts of this question deal with the differential equation $y' = x + y$.

(a) Verify that $y = -x - 1$ is a solution of this differential equation.
Solution: With $y = -x - 1$, we have $y' = \frac{dy}{dx} = -1$. On the right side of the equation $x + y = x + (-x - 1) = -1$. Therefore $y = -x - 1$ solves this differential equation.

(b) Sketch the direction field of this equation by hand, showing the slopes at all points with $-3 \leq x \leq 3, -3 \leq y \leq -3$ with integer coordinates.
Solution: (See plot with part (c) below).

(c) On your direction field graph, sketch approximate graphs of the solutions of this differential equation satisfying the initial conditions $y(0) = -2$, $y(0) = 0$, $y(0) = 1$.
Solution: Here is a Maple plot showing this direction field and the solutions.



Note that more direction field segments are plotted here than the problem asked for, because Maple is using a default grid that is finer than the grid of points with integer coordinates.

(d) Use Euler's method with $n = 5$, $\Delta x = .2$ to approximate the solution of the equation with the initial condition $y(0) = 1$ for $0 \leq x \leq 1$.

Solution: We have

$$\begin{array}{ll}
x_0 = 0 & y_0 = 1 \quad \text{given from initial condition} \\
x_1 = .2 & y_1 = y_0 + (x_0 + y_0)\Delta x = 1 + 1(.2) = 1.2 \\
x_2 = .4 & y_2 = y_1 + (x_1 + y_1)\Delta x = 1.2 + (1.4)(.2) = 1.48 \\
x_3 = .6 & y_3 = y_2 + (x_2 + y_2)\Delta x = 1.48 + (1.88)(.2) = 1.856 \\
x_4 = .8 & y_4 = y_3 + (x_3 + y_3)\Delta x = 1.856 + (2.456)(.2) = 2.3472 \\
x_5 = 1.0 & y_5 = y_4 + (x_4 + y_4)\Delta x = 2.3472 + (3.1472)(.2) = 2.97664
\end{array}$$

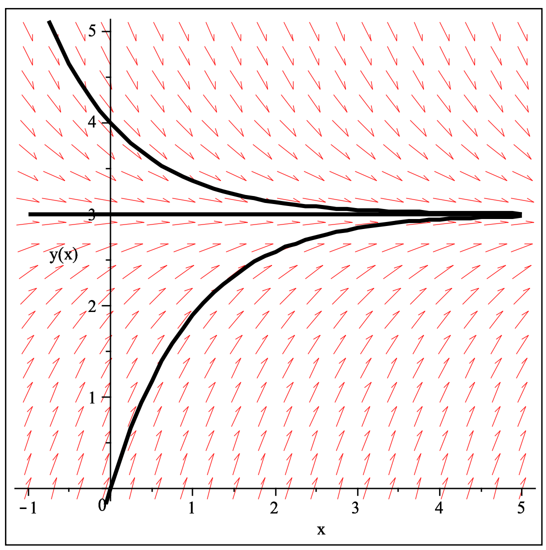
2. All parts of this question deal with the differential equation $y' = 3 - y$.

   (a) Sketch the direction field of this equation by hand, showing the slopes at all points with $-1 \le x \le 5, 0 \le y \le 5$ with integer coordinates.

   Solution: See plot with part (b) below. Note again that Maple is using a finer default grid to draw the direction field.

   (b) On your direction field graph, sketch approximate graphs of the solutions of this differential equation satisfying the initial conditions $y(0) = 0$, $y(0) = 3$, and $y(0) = 4$. What is the long term behavior of each?

   Solution:



   From the graph, we see that $\lim_{x \to +\infty} y(x) = 3$ for all three solutions.

   (c) Use Euler's method with $n = 5$, $\Delta x = .2$ to approximate the solution of the equation with the initial condition $y(0) = 1$ for $0 \le x \le 1$.

Solution: We have

$$
\begin{aligned}
x_0 &= 0 & y_0 &= 1 \quad \text{given from initial condition} \\
x_1 &= .2 & y_1 &= y_0 + (3 - y_0)\Delta x = 1.4 \\
x_2 &= .4 & y_2 &= y_1 + (3 - y_1)\Delta x = 1.72 \\
x_3 &= .6 & y_3 &= y_2 + (3 - y_2)\Delta x = 1.976 \\
x_4 &= .8 & y_4 &= y_3 + (3 - y_3)\Delta x = 2.1808 \\
x_5 &= 1.0 & y_5 &= y_4 + (3 - y_4)\Delta x = 2.34464
\end{aligned}
$$

(d) This is a separable equation. Derive the general solution by separating variables and integrating.

Solution: Separating variables and integrating yields:

$$
\begin{aligned}
\frac{dy}{3-y} &= dx \\
\int \frac{dy}{3-y} &= \int dx \\
-\ln|3-y| &= x + c \\
3 - y &= be^{-x} \text{ (where } b = \pm e^{-c}) \\
y &= 3 - be^{-x}.
\end{aligned}
$$

(e) What is the exact formula for the solution of the differential equation with the initial condition $y(0) = 1$?

Solution: Using the answer from part (d), we solve for b from the initial condition:

$$1 = 3 - be^0 \Rightarrow b = 2.$$

So the solution with this initial condition is $y = 3 - 2e^{-x}$.

3. Solve each of the following differential equations, finding an explicit formula for y as a function of x if possible (you can leave the solution in implicit form if not). In some cases you are given initial conditions; use them to determine the arbitrary constant in the general solution.

(a) $y' = \dfrac{\tan(y)}{x^2 + 4x + 3}$, find the general solution.

Solution: Separate variables and integrate, using partial fractions on the right and

$\tan(y) = \dfrac{\sin(y)}{\cos(y)}$ on the left:

$$\begin{aligned}
\frac{1}{\tan(y)}dy &= \frac{1}{x^2+4x+3}\ dx \\
\int \frac{\cos(y)}{\sin(y)}\ dy &= \int \frac{1/2}{x+1} + \frac{-1/2}{x+3}\ dx \\
\ln|\sin(y)| &= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + c \\
&= \ln\sqrt{\left|\frac{x+1}{x+3}\right|} + c \quad \text{(by properties of ln)} \\
\sin(y) &= b\cdot\sqrt{\left|\frac{x+1}{x+3}\right|} \qquad \text{(where } b = \pm e^c\text{)} \\
y &= \sin^{-1}\left(b\cdot\sqrt{\left|\frac{x+1}{x+3}\right|}\right).
\end{aligned}$$

(b) $y' = \dfrac{(x-1)y^5}{x^2(2y^3-y)}$, find the general solution.

Solution: Separate variables, simplify, and integrate by the power rule,

$$\begin{aligned}
\frac{2y^3-y}{y^5}\ dy &= \frac{x-1}{x^2}\ dx \\
2y^{-2} - y^{-4}\ dy &= x^{-1} - x^{-2}\ dx \\
\int 2y^{-2} - y^{-4}\ dy &= \int x^{-1} - x^{-2}\ dx \\
-2y^{-1} + \frac{1}{3}y^{-3} &= \ln|x| + x^{-1} + c.
\end{aligned}$$

(This is one where the solution is defined implicitly by the equation.)

(c) $y' = 2xy$ with $y(0) = 4$.

Solution: Separate variables, and integrate,

$$\begin{aligned}
\frac{1}{y}\ dy &= 2x\ dx \\
\ln|y| &= x^2 + c \\
y &= be^{x^2} \quad \text{(where } b = \pm e^c\text{)}.
\end{aligned}$$

From the initial condition $4 = be^0$, so $b = 4$. The solution is $y = 4e^{x^2}$.

(d) $y' = x(y^2+1)$ with $y(0) = 1$.

Solution: Separate variables, and integrate,

$$\begin{aligned}\frac{1}{y^2+1}\,dy &= x\,dx\\ \tan^{-1}(y) &= \frac{x^2}{2}+c\\ y &= \tan\left(\frac{x^2}{2}+c\right).\end{aligned}$$

From the initial condition $1 = \tan(0+c)$, so $c = \frac{\pi}{4}$ gives one solution, and $y = \tan\left(\frac{x^2}{2}+\frac{\pi}{4}\right)$. (Note that in this case the periodicity of the tangent function means that other values of c with $\tan(c) = 1$ will define equivalent functions.)

(e) $y' = 2xy^2 + 3x^2y^2$ with $y(1) = -1$.

Solution: To see that this equation is separable, we factor the right hand side: $y' = (2x+3x^2)y^2$ which is of the form $g(x)h(y)$ as required. Then we separate variables and integrate as before.

$$\begin{aligned}\int y^{-2}\,dy &= \int 2x+3x^2\,dx\\ -\frac{1}{y} &= x^2+x^3+c\\ y &= \frac{-1}{x^2+x^3+c}.\end{aligned}$$

From the initial condition $-1 = \dfrac{-1}{2+c}$ so $c = -1$. The solution is

$$y = \frac{-1}{x^2+x^3-1}.$$

(f) $y' = 3y(1-y/20)$ with $y(0) = 5$. (Note: this is a logistic equation.)

Solution: By the general form of the solutions of logistic equations derived in class, we have $M = 20$ and $k = 3$, so

$$y = \frac{20}{1+ae^{-3x}}$$

for some constant a. To determine a, we use the initial condition: $5 = \dfrac{20}{1+ae^0}$, so $1+a = 4$ and $a = 3$. The solution is

$$y = \frac{20}{1+3e^{-3x}}.$$

4. (Radiocarbon dating.) C^{14} is a radioactive isotope of carbon with half-life of approximately 5700 years. It occurs naturally and is incorporated in living tissues by normal

metabolic processes. Assume that any C^{14} present when the tissue dies decays exponentially from that time on. Carbon extracted from an ancient skull contains only one-sixth as much C^{14} as carbon extracted from present-day bone. How old is the skull?

Solution: Since the C^{14} is decaying exponentially, writing Q for the amount, we have $Q' = kQ$ for some negative constant k. The solutions of this differential equation are

$$Q(t) = Q(0)e^{kt}.$$

Since the half-life is 5700 years, whatever $Q(0)$ was,

$$Q(5700) = Q(0)e^{5700k} = \frac{1}{2}Q(0).$$

Therefore solving the equation $e^{5700k} = \frac{1}{2}$ for k, we have $k = \frac{\ln(1/2)}{5700} \doteq -0.1216 \times 10^{-3}$. Now thinking of $Q(0)$ as the amount of C^{14} that was in the skull at the time the animal it came from died, in the time elapsed since death to the present, the amount present has decayed to $\frac{1}{6}Q(0)$. Hence we want to solve the equation

$$\frac{1}{6}Q(0) = Q(0)e^{(-.1216\times 10^{-3})t}$$

for t to find the age:

$$t = \frac{\ln(1/6)}{-.1216 \times 10^{-3}} \doteq 14375 \text{ years.}$$

5. (Newton's Law of Heating/Cooling.) A pitcher of buttermilk initially at 25° C is cooled by setting it out on the front porch on a cold day when the exterior temperature is a constant 0° C. Suppose that the buttermilk has reached 15° C after 20 minutes.

   (a) What will the temperature be after 25 minutes?

   (b) When will the temperature reach 5° C?

   Solution: For both parts, we use the differential equation $T' = k(T - A)$ from Newton's Law. The solutions are $T = A + be^{kt}$. Now in the situation of the problem, $A = 0$ (the ambient temperature on the porch). We have $T(0) = 25$, so $25 = 0 + be^0$ which shows $b = 25$. In addition from $T(20) = 15$, $15 = 25e^{20k}$ so $k = \frac{\ln(15/25)}{20} = \ln(.6)/20 \doteq -.02554$. So the temperature at all later times is

   $$T(t) = 25e^{(-0.02554)t}.$$

   (a) After 25 minutes $T(25) = 25e^{(-0.02554)(25)} \doteq 13.2°$ C.

   (b) The temperature reaches 5° C when $5 = 25e^{(-0.02554)t}$, so $t = \frac{\ln(.2)}{-0.02554} \doteq 63$ minutes.

6. The population of fish in a lake is attacked by a microscopic water-borne parasite at $t = 0$, and as a result the population declines at a rate proportional to the *square root* of the population from that time on.

(a) Express this statement about the rate of growth of the population P as a differential equation. (There should be a constant of proportionality, say $-k$, in your equation.)

Solution: The differential equation is

$$P' = \frac{dP}{dt} = -k\sqrt{P}.$$

(We write the constant of proportionality as $-k$ here because we know that since P is decreasing over time, $P' < 0$. The value of k will be positive with this choice.)

(b) Use separation of variables to find the general solution of your differential equation.

Solution:

$$\begin{aligned} \int \frac{dP}{\sqrt{P}} &= \int -k \; dt \\ 2P^{1/2} &= -kt + c \\ P &= \left(\frac{-k}{2} t + \frac{c}{2} \right)^2 . \end{aligned}$$

(Note that $\frac{c}{2}$ is just another arbitrary constant, though.)

(c) At $t = 0$ there were 900 fish in the lake; 441 were left after 6 weeks. When did the fish population disappear entirely?

Solution: We take t as the time *in weeks* for simplicity. (The only differences if we used a different time unit would be in the values of k and c.) From the answer to part (b), $900 = \left(0 + \frac{c}{2}\right)^2$ so $\frac{c}{2} = 30$. Thus $P = \left(\frac{-k}{2}t + 30\right)^2$. When $t = 6$, $441 = (-3k + 30)^2$, so $21 = -3k + 30$ and $k = 3$. The population as a function of time is

$$P = \left(\frac{-3}{2} t + 30 \right)^2 .$$

The population reaches $P = 0$ when $\frac{-3}{2}t + 30 = 0$, so $t = 20$ weeks.