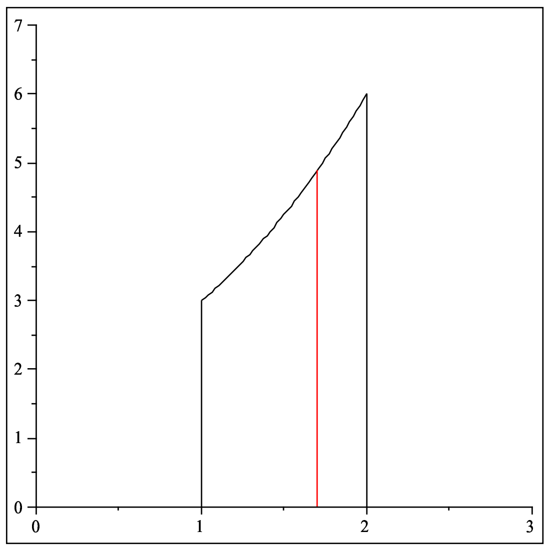
Math 132: Calculus for Physical and Life Sciences 2
Solutions for Problem Set 6
Due Friday, March 14, 2008, at the beginning of class.

Note: To save space, only a few of the graphs are shown.

1. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, and a typical disk or washer.

   (a) $y = x^2 + 2$, $x = 1$, $x = 2$, $y = 0$; about the x-axis

   Solution: In the following plot we show the region being rotated and a typical slice through the region by a plane perpendicular to the x-axis (shown in red). The red curve generates the cross-section, which is a disk in this case.

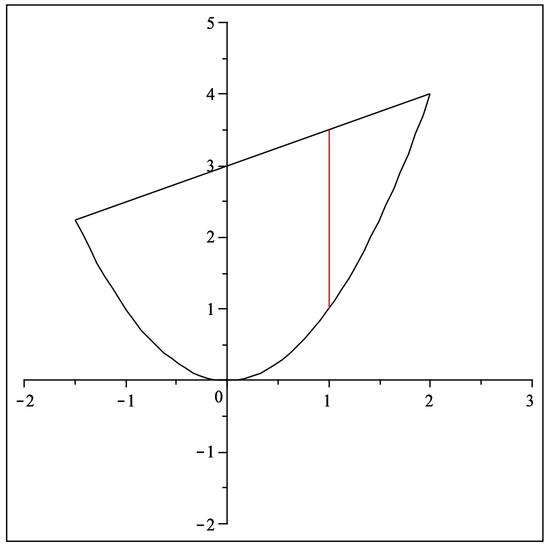


   The cross-section by the plane $x = c$ perpendicular to the x-axis is a disk with radius $x^2 + 2$ and area $\pi(x^2 + 2)^2$. The volume is

$$\begin{aligned} V &= \int_1^2 \pi(x^2+2)^2 \, dx \\ &= \pi \int_1^2 x^4 + 4x^2 + 4 \, dx \\ &= \pi \left(\frac{x^5}{5} + \frac{4x^3}{3} + 4x \Big|_1^2 \right) \\ &= \pi \left[\left(\frac{32}{5} + \frac{32}{3} + 8 \right) - \left(\frac{1}{5} + \frac{4}{3} + 4 \right) \right] \\ &= \frac{293\pi}{15}. \end{aligned}$$

(b) $y = \frac{x}{2} + 3$, $y = x^2$; about the x-axis

Solution: The line $y = \frac{x}{2} + 3$ crosses the parabola $y = x^2$ at $x = -\frac{3}{2}, 2$. The cross-section by a plane $x = c$ is a washer with inner radius x^2 and outer radius $\frac{x}{2}+3$ (except for one disk at $c = 0$ where the hole "closes up" for a single x-value).



The volume is

$$\begin{aligned} V &= \int_{-3/2}^{2} \pi \left(\frac{x}{2} + 3\right)^2 - \pi (x^2)^2 \, dx \\ &= \pi \int_{-3/2}^{2} \frac{x^2}{4} + 3x + 9 - x^4 \, dx \\ &= \pi \left(\frac{x^3}{12} + \frac{3x^2}{2} + 9x - \frac{x^5}{5} \Big|_{-3/2}^{2} \right) \\ &= \pi \left(\frac{274}{15} - \frac{-711}{80} \right) \\ &= \frac{6517\pi}{240}. \end{aligned}$$

(c) $y = \sin x$, $x = 0$, $x = \pi$, $y = 0$; about the line $y = 2$.

Solution: The cross-section by a plane $x = c$ is a washer with inner radius $2 - \sin(x)$

and outer radius 2. The volume is

$$\begin{aligned} V &= \int_0^{\pi} 4\pi - \pi(2-\sin(x))^2 \, dx \\ &= \pi \int_0^{\pi} 4\sin(x) - \sin^2(x) \, dx \\ &= \pi \left(-4\cos(x) - \frac{x}{2} + \frac{1}{4}\sin(2x) \Big|_0^{\pi} \right) \\ &= \pi \left(4 - \frac{\pi}{2} + 4 \right) \\ &= 8\pi - \frac{\pi^2}{2}. \end{aligned}$$

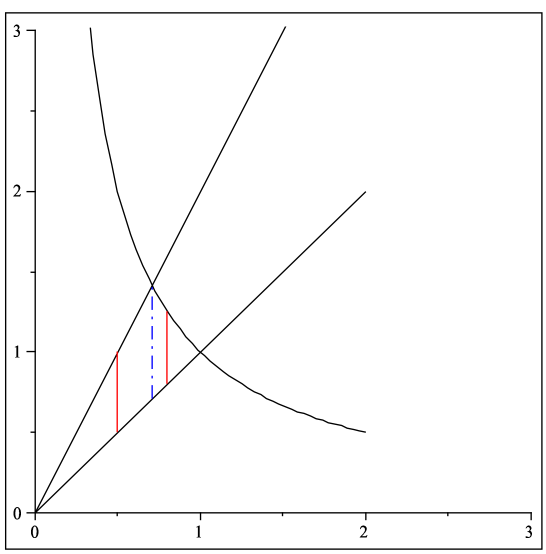
(d) $y = x^3$, $y = \sqrt{x}$, $x = 0$, $x = 1$; about the x-axis

Solution: The cross-section by a plane $x = c$ is a washer with outer radius $\sqrt{x}$ and inner radius x^3. The volume is

$$\begin{aligned} V &= \int_0^1 \pi(\sqrt{x})^2 - \pi(x^3)^2 \, dx \\ &= \pi \int_0^1 x - x^6 \, dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^7}{7} \Big|_0^1 \right) \\ &= \frac{5\pi}{14}. \end{aligned}$$

(e) $y = \frac{1}{x}$, $y = x$, $y = 2x$ for $x > 0$; about the x-axis

Solution: Between $x = 0$ and $x = \sqrt{2}/2$, the top edge of the region is the line $y = 2x$ and the bottom is the line $y = x$. From $x = \sqrt{2}/2$ to $x = 1$, the top edge is $y = \frac{1}{x}$ and the bottom is the line $y = x$.

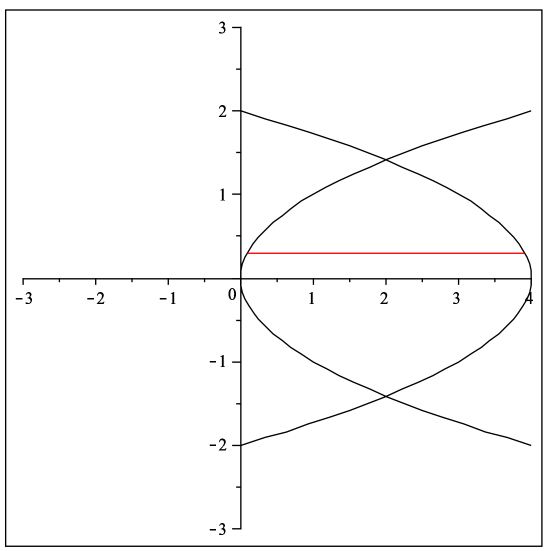


The solid generated rotating around the x-axis is the union of two solids meeting only along the plane $x = \sqrt{2}/2$. The cross-sections of the first are the washers with inner radius x and outer radius $2x$ given by rotating the slice to the left of the blue dashed line in the figure. The cross-sections of the second are also washers with inner radius x and outer radius $1/x$ obtained by rotating the slices to the right of the blue dashed line. Hence the total volume is

$$\begin{aligned} V &= \int_0^{\sqrt{2}/2} \pi(2x)^2 - \pi x^2\, dx + \int_{\sqrt{2}/2}^1 \pi(1/x)^2 - \pi x^2\, dx \\ &= \pi\left(\int_0^{\sqrt{2}/2} 3x^2\, dx + \int_{\sqrt{2}/2}^1 (1/x)^2 - x^2\, dx\right) \\ &= \pi\left(\left(x^3\Big|_0^{\sqrt{2}/2}\right) + \left(\frac{-1}{x} - \frac{x^3}{3}\Big|_{\sqrt{2}/2}^1\right)\right) \\ &= \frac{4\pi}{3}(\sqrt{2}-1). \end{aligned}$$

(f) $x = 4 - y^2$, $x = y^2$; about the y-axis

Solution: The solid will be generated by rotating the region between these two parabolas with $x \geq 0$:



Since we rotate around the y-axis, the slices of interest are the cross-sections by planes $y = d$. These are washers (except for the cross-section at $y = 0$ where the hole in the center "closes up"). The inner radius is y^2 and the outer radius is

$4 - y^2$. The parabolas cross at $y = \pm\sqrt{2}$. The volume is

$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi(4-y^2)^2 - \pi(y^2)^2 \, dy \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} 16 - 8y^2 \, dy \\ &= \pi \left(16y - \frac{8y^3}{3} \bigg|_{-\sqrt{2}}^{\sqrt{2}} \right) \\ &= \frac{64\pi\sqrt{2}}{3}. \end{aligned}$$

(g) $x = \frac{1}{y}$, $y = 4$, $x = 2$; about the y-axis

Solution: The cross-section by a plane $y = d$ is a washer with inner radius $\frac{1}{y}$ and outer radius 2. The lower limit of integration comes from where $x = 2$ crosses $x = 1/y$. So the volume is

$$\begin{aligned} V &= \int_{1/2}^{4} 4\pi - \pi(1/y)^2 \, dy \\ &= \pi \int_{1/2}^{4} 4 - \frac{1}{y^2} \, dy \\ &= \pi \left(4y + \frac{1}{y} \bigg|_{1/2}^{4} \right) \\ &= \frac{49\pi}{4}. \end{aligned}$$

2. The base of a certain solid is the circle $x^2 + y^2 = 9$ and each cross section perpendicular to the x-axis is an equilateral triangle with one side across the base. Find the volume of the solid.

Solution: The area of an equilateral triangle of side s is

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(s)\left(\frac{s\sqrt{3}}{2}\right) = \frac{s^2\sqrt{3}}{4}.$$

For this solid, in the cross-section at $x = c$ will have $s = 2\sqrt{9 - x^2}$, so the volume is

$$\begin{aligned} V &= \int_{-3}^{3} A(x)\ dx \\ &= \int_{-3}^{3} \frac{4(9 - x^2)\sqrt{3}}{4}\ dx \\ &= \sqrt{3}\int_{-3}^{3} 9 - x^2\ dx \\ &= \sqrt{3}\left(9x - \frac{x^3}{3}\Big|_{-3}^{3}\right) \\ &= 36\sqrt{3}. \end{aligned}$$

3. A nose cone for a space reentry vehicle is designed so that a cross section, taken x feet from the the tip and perpendicular to the axis of symmetry, is a circle of radius $\frac{1}{4}x^2$ ft. Find the volume of the nose cone given that its length is 20 ft.

   Taking the tip as $x = 0$ and the base $x = 20$, we have

$$\begin{aligned} V &= \int_{0}^{20} A(x)\ dx \\ &= \int_{0}^{20} \pi\left(\frac{1}{4}x^2\right)^2\ dx \\ &= \frac{\pi}{16}\int_{0}^{20} x^4\ dx \\ &= \frac{\pi}{16}\left(\frac{x^5}{5}\Big|_{0}^{20}\right) \\ &= 40000\pi. \end{aligned}$$

   (units are cubic feet).

4. Graph the curve and find its exact length.

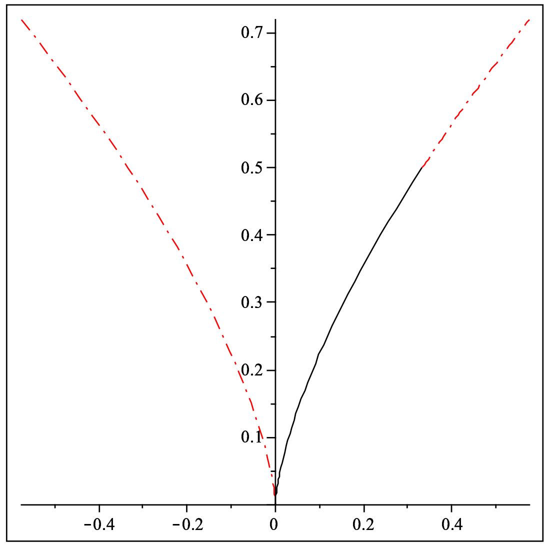
   (a) $x = 4t + 3$, $y = 3t - 2$, $0 \le t \le 2$

   Solution: The graph is a straight line from the point $(3, -2)$ to the point $(11, 4)$. The arclength is

$$\begin{aligned} L &= \int_{0}^{2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \\ &= \int_{0}^{2} \sqrt{16 + 9}\ dt \\ &= 10. \end{aligned}$$

(b) $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$, $0 \le t \le 1$

Solution: The curve here is the portion of a "cuspidal cubic" in the plane shown in black here:



The arclength is

$$
\begin{aligned}
L &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^1 \sqrt{(t^2)^2 + t^2}\, dt \\
&= \int_0^1 t\sqrt{t^2+1}\, dt \text{ let } u = t^2+1,\ du = 2t\, dt \\
&= \int_{u=1}^{u=2} \frac{1}{2}\sqrt{u}\, du \\
&= \left(\frac{1}{3}u^{3/2}\Big|_1^2\right) \\
&= \frac{1}{3}(2\sqrt{2}-1).
\end{aligned}
$$

(c) $x = 3\sin(2t)$ and $y = 3\cos(2t)$, $0 \le t \le \frac{\pi}{2}$

Solution: The curve is the arc of a circle with radius 3 and center $(0,0)$ starting at $(0,3)$ and proceeding clockwise to $(0,-3)$ (one half of the full circle). The arclength is

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi/2} \sqrt{36\cos^2(2t) + 36\sin^2(2t)}\, dt \\ &= \int_0^{\pi/2} 6\, dt \\ &= 3\pi. \end{aligned}$$

where we used the trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to simplify before integrating.

(d) $y = 3x^2 + 2$ and $2 \le x \le 4$

The curve is a segment of a parabola opening up. The parameter is x so the arclength is

$$\begin{aligned} L &= \int_2^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_2^4 \sqrt{1 + 36x^2}\, dx \qquad \text{so let } u = 6x,\ du = 6\, dx \\ &= \frac{1}{6}\int_{u=12}^{u=24} \sqrt{1 + u^2}\, du \text{ which is the form \#21 in the table} \\ &= \frac{1}{6}\left(\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left|u + \sqrt{1+u^2}\right|\Big|_{12}^{24}\right) \\ &= 2\sqrt{577} - \sqrt{145} + \frac{1}{12}\ln\left|24 + \sqrt{577}\right| - \frac{1}{12}\ln\left|12 + \sqrt{145}\right|. \end{aligned}$$

5. Use Simpson's Rule with $n = 10$ to estimate the arc length of the curve $y = \ln x$ for $1 \le x \le 4$.

Solution: The length is given by the integral

$$L = \int_1^4 \sqrt{1 + \frac{1}{x^2}}\, dx.$$

Let $f(x) = \sqrt{1 + \frac{1}{x^2}}$. For Simpson with $n = 10$, $\Delta x = \frac{3}{10}$ and we obtain the approximation

$$\begin{aligned} L &\doteq \frac{(3/10)}{3}\left(f(1) + 4f\left(\frac{13}{10}\right) + 2f\left(\frac{16}{10}\right) + 4f\left(\frac{19}{10}\right) + 2f\left(\frac{22}{10}\right) + 4f\left(\frac{25}{10}\right) + \right. \\ &\quad \left. 2f\left(\frac{28}{10}\right) + 4f\left(\frac{31}{10}\right) + 2f\left(\frac{34}{10}\right) + 4f\left(\frac{37}{10}\right) + f(4)\right) \\ &\doteq 3.343018. \end{aligned}$$

(This approximate value is correct to 3 decimal places if we round – the exact value is about 3.342799189.)

6. Find the average value of the function on the given interval.

(a) $y = x\cos(3x)$, $[0, \pi]$

Solution: By the definition, the average value is

$$f_{ave} = \frac{1}{\pi}\int_0^{\pi} x\cos(3x)\ dx.$$

To evaluate this, we use parts with $u = x$, $dv = \cos(3x)\ dx$.

$$\begin{aligned} f_{ave} &= \frac{1}{\pi}\int_0^{\pi} x\cos(3x)\ dx \\ &= \frac{1}{\pi}\left(\frac{x}{3}\sin(3x)\Big|_0^{\pi} - \frac{1}{3}\int_0^{\pi}\sin(3x)\ dx\right) \\ &= \frac{1}{\pi}\left(0 + \frac{1}{9}\cos(3x)\Big|_0^{\pi}\right) \\ &= \frac{-2}{9\pi}. \end{aligned}$$

(b) $y = x\sqrt{1+3x^2}$, $[1, 3]$

Solution: By the definition, the average value is

$$f_{ave} = \frac{1}{2}\int_1^3 x\sqrt{1+3x^2}\ dx.$$

To evaluate this, we use the u-substitution $u = 1 + 3x^2$, so $du = 6x\,dx$.

$$\begin{aligned} f_{ave} &= \frac{1}{2}\int_1^3 x\sqrt{1+3x^2}\ dx \\ &= \frac{1}{2}\left(\int_{u=4}^{u=28}\frac{1}{6}u^{1/2}\ du\right) \\ &= \frac{1}{18}\left(u^{3/2}\Big|_4^{28}\right) \\ &= \frac{1}{9}(28\sqrt{7} - 4). \end{aligned}$$

7. Find the number b such that the average value of $f(x) = x^3 - 1$ on the interval $[0, b]$ is equal to 6.

Solution: We want b such that

$$
\begin{aligned}
6 &= \frac{1}{b}\int_0^b x^3 - 1\,dx \\
&= \frac{1}{b}\left(\frac{x^4}{4} - x\Big|_0^b\right) \\
&= \frac{b^3}{4} - 1.
\end{aligned}
$$

This is true for $b = \sqrt[3]{28} \doteq 3.04$.

8. The amount of a certain drug present in a patient's body for the first 4 days after the drug has been administered is

$$C(t) = 5e^{-0.2t}$$

units. Determine the average amount of the drug present in the patient's body for the first 4 days after the drug has been administered.

Solution: The average amount is

$$
\begin{aligned}
C_{ave} &= \frac{1}{4}\int_0^4 5e^{(-0.2)t}\,dt \\
&= \frac{1}{4}\left(-25e^{(-0.2)t}\Big|_0^4\right) \\
&= \frac{25}{4}\left(1 - e^{-0.8}\right) \\
&\doteq 3.44\,\text{units}.
\end{aligned}
$$