College of the Holy Cross, Spring 2008
Math 132, Solutions for Midterm Exam 2 (All Sections)
Wednesday, March 26, 6 PM

I.

A. (5) Use midpoint Riemann sums with $n = 4$ to approximate $\int_1^2 \sqrt{1 + \ln(x)}\, dx$.

Solution. On the interval $[1, 2]$, with $n = 4$, we have $\Delta x = \frac{2-1}{4} = .25$, so $x_0 = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75, x_4 = 2$, and the midpoints are $m_1 = 1.125, m_2 = 1.375, m_3 = 1.625, m_4 = 1.875$. The midpoint Riemann sum is

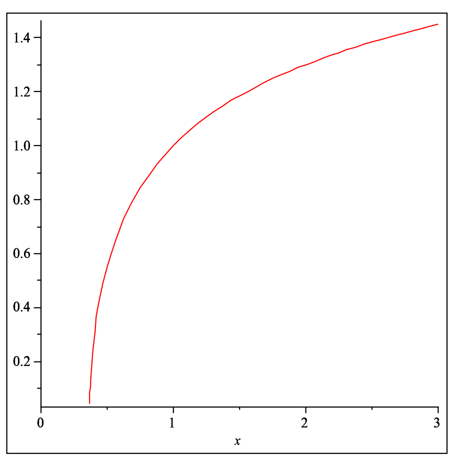
$$\begin{aligned} M_4 &= (.25)\left(\sqrt{1+\ln(1.125)} + \sqrt{1+\ln(1.375)} + \sqrt{1+\ln(1.625)} + \sqrt{1+\ln(1.875)}\right) \\ &\doteq (.25)(1.057252589) + (1.148239405) + (1.218814102) + (1.276169526)) \\ &= 1.175118905. \end{aligned}$$

B. (5) Use the trapezoidal rule with $n = 4$ to approximate $\int_1^2 \sqrt{1 + \ln(x)}\, dx$.

Solution. By the trapezoidal rule formula, writing $f(x) = \sqrt{1 + \ln(x)}$,

$$\begin{aligned} T_4 &= \frac{(.25)}{2}(f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)) \\ &\doteq \frac{(.25)}{2}(1. + 2(1.105958205) + 2(1.185523137) + 2(1.248845782) + 1.301209891) \\ &= 1.172733017. \end{aligned}$$

C. (5) The function $y = \sqrt{1 + \ln(x)}$ is plotted below.



Solution. Since the graph is concave down on $[1,2]$,

The midpoint approximation is a overestimate.

The trapezoidal rule approximation is an underestimate.

II. For each of the following improper integrals, set up and evaluate the appropriate limits to determine whether the integral converges. If so, find its value; if not, say "does not converge." (Credit will be given only for the correct limit calculation.)

A. (8) $\int_1^{\infty} e^{-x}\, dx$

Solution. The integral is improper because the interval of integration is infinite.

$$\begin{aligned}\int_1^{\infty} e^{-x}\, dx &= \lim_{b\to\infty}\int_1^b e^{-x}\, dx \\ &= \lim_{b\to\infty}\left(-e^{-x}\Big|_1^b\right) \\ &= \lim_{b\to\infty}\frac{1}{e}-\frac{1}{e^b} \\ &= \frac{1}{e}.\end{aligned}$$

The integral converges; the value is $\frac{1}{e}$.

B. (8) $\int_{-1}^{1}\frac{1}{x^2}\, dx$

Solution. The integral is improper because the integrand has an infinite discontinuity at $x=0$ in the interior of the interval. Therefore,

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^2}\, dx &= \lim_{b\to 0^-}\int_{-1}^{b}\frac{1}{x^2}\, dx+\lim_{a\to 0^+}\int_a^1\frac{1}{x^2}\, dx \\ &= \lim_{b\to 0^-}\left(-\frac{1}{x}\Big|_{-1}^{b}\right)+\lim_{a\to 0^+}\left(-\frac{1}{x}\Big|_a^1\right) \\ &= \lim_{b\to 0^-}\left(-\frac{1}{b}-1\right)+\lim_{a\to 0^+}\left(-1+\frac{1}{a}\right).\end{aligned}$$

Neither limit exists, so this improper integral *diverges.*

C. (9) $\int_0^1 x\ln(x)\, dx.$

Solution. This integral is improper because of the infinite discontinuity of $\ln(x)$ at $x = 0$.

$$\int_0^1 x\ln(x)\ dx = \lim_{a\to 0^+}\int_a^1 x\ln(x)\ dx$$

(if the limit exists). To integrate, we use parts letting $u = \ln(x)$ and $dv = x\ dx$, so $du = \frac{1}{x}\ dx$ and $v = \frac{x^2}{2}$. Hence

$$\begin{aligned} &= \lim_{a\to 0^+}\left[\left(\frac{x^2}{2}\ln(x)\Big|_a^1\right) - \int_a^1 \frac{x^2}{2}\cdot\frac{1}{x}\ dx\right] \\ &= \lim_{a\to 0^+}\left[-\frac{a^2}{2}\ln(a) - \frac{1}{4} + \frac{a^2}{4}\right]. \end{aligned}$$

To evaluate the limit of the first term here, we note that this is an indeterminate form $0\cdot\infty$. Hence we will use L'Hopital's Rule, after rearranging:

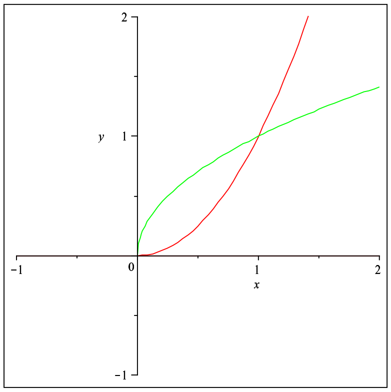
$$\begin{aligned} \lim_{a\to 0^+}\frac{a^2}{2}\ln(a) &= \lim_{a\to 0^+}\frac{\ln(a)}{\frac{2}{a^2}} \\ &= \lim_{a\to 0^+}\frac{\frac{1}{a}}{\frac{-4}{a^3}} \\ &= \lim_{a\to 0^+}\frac{a^2}{-4} \\ &= 0. \end{aligned}$$

Hence the integral converges to $\frac{-1}{4}$ since the last term on the right above also tends to 0 as $a \to 0^+$.

III. The region R is bounded by the graphs $y = \sqrt{x}$ and $y = x^2$.

A. (5) Sketch the region R.

Solution. The upper curve (in green) is $y = \sqrt{x}$. The lower (in red) is $y = x^2$. The two graphs cross when $x^{1/2} = x^2$, so $x = x^4$. This gives $x = 0$ and $x = 1$. The region is shown here:



B. (5) Set up and compute an integral to find the area of R.

Solution. The area of R is

$$\begin{aligned}\int_0^1 \sqrt{x} - x^2 \, dx &= \left(\frac{2x^{3/2}}{3} - \frac{x^3}{3} \Big|_0^1 \right) \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3}.\end{aligned}$$

C. (10) The region R is rotated about the x-axis to generate a solid. Set up and compute an integral to find its volume.

Solution. The cross-sections by planes perpendicular to the x-axis are washers with outer radius $r_{out} = \sqrt{x}$ and inner radius $r_{in} = x^2$. The volume is the integral of the cross-section area, so

$$\begin{aligned}V &= \int_0^1 \pi(\sqrt{x})^2 - \pi(x^2)^2 \, dx \\ &= \pi \int_0^1 x - x^4 \, dx \\ &= \pi \left(\frac{x}{2} - \frac{x^5}{5} \Big|_0^1 \right) \\ &= \frac{3\pi}{10}.\end{aligned}$$

D. (10) The region R is rotated about the y-axis to generate a solid. Set up and compute an integral to find its volume.

Solution. On the basis of symmetry of the region R, we can see that this volume will be the same as the volume in part C. To compute this via an integral, we need to set up in terms of y. The graph $y = x^2$ is $x = \sqrt{y}$ and the graph $y = \sqrt{x}$ is $x = y^2$. The cross-sections by planes perpendicular to the y-axis are washers with outer radius $r_{out} = \sqrt{y}$ and inner radius $r_{in} = y^2$. The volume is the integral of the cross-section area, so

$$V = \int_0^1 \pi(\sqrt{y})^2 - \pi(y^2)^2 \, dy = \frac{3\pi}{10}$$

by the same computations as in part C.

IV. (15) Set up and compute the integral for the arclength of the curve $x = 5t^2 + 2, y = t + 1$, $0 \le t \le 2$. (You may consult the table of integrals for this one; if you do, say which table entry you are using.)

Solution. The arclength is

$$\begin{aligned} L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^2 \sqrt{(10t)^2 + 1}\, dt. \end{aligned}$$

Let $u = 10t$, then $dt = \frac{1}{10}du$, so

$$L = \frac{1}{10}\int_0^{20} \sqrt{u^2+1}\, du.$$

This is the form from #21 in the table of integrals, with $a = 1$. Hence

$$\begin{aligned} L &= \frac{1}{10}\left(\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln(u + \sqrt{u^2+1})\Big|_0^{20}\right) \\ &= \sqrt{401} + \frac{1}{20}\ln(20 + \sqrt{401}) \\ &\doteq 20.21. \end{aligned}$$

V. (15) Find the average value of $f(x) = \dfrac{x^3}{\sqrt{25 - x^2}}$ on the interval $0 \le x \le 2$. (Use a trigonometric substitution to compute the integral you need.)

Solution. The average value is computed by the integral

$$f_{ave} = \frac{1}{2-0}\int_0^2 \frac{x^3}{\sqrt{25-x^2}}\, dx.$$

To evaluate the integral, we use the trigonometric substitution $x = 5\sin\theta$, so $dx = 5\cos\theta\, d\theta$ and $\sqrt{25 - x^2} = \sqrt{25 - 25\sin^2\theta} = 5\sqrt{1 - \sin^2\theta} = 5\cos\theta$. This gives

$$\begin{aligned} \int \frac{x^3}{\sqrt{25-x^2}}\, dx &= \int \frac{125\sin^3\theta}{5\cos\theta} \cdot 5\cos\theta\, d\theta \\ &= 125\int \sin^3\theta\, d\theta \\ &= 125\int \sin^2\theta \cdot \sin\theta\, d\theta \\ &= 125\int (1 - \cos^2\theta)\sin\theta\, d\theta \\ &= 125\int \sin\theta\, d\theta - 125\int \cos^2\theta \sin\theta\, d\theta \\ &= -125\cos\theta + \frac{125}{3}\cos^3\theta + C. \end{aligned}$$

Therefore since $\cos\theta = \frac{\sqrt{25-x^2}}{5}$ from the reference triangle (opposite $= x$, hypotenuse $= 5$, adjacent $= \sqrt{25-x^2}$), we have

$$\begin{aligned} f_{ave} &= \frac{1}{2}\left(-25\sqrt{25-x^2} + \frac{1}{3}(25-x^2)^{3/2} \Big|_0^2 \right) \\ &= \frac{1}{2}\left(-25\sqrt{21} + 7\sqrt{21} + 125 - \frac{125}{3} \right) \\ &= \frac{1}{2}\left(\frac{250}{3} - 18\sqrt{21} \right) \\ &= \frac{125}{3} - 9\sqrt{21} \\ &\doteq 0.42348541. \end{aligned}$$