College of the Holy Cross, Spring 2008
Math 132, Solutions for Midterm Exam 1 (All Sections)
Wednesday, February 20

I. (10) State the definition of the definite integral $\int_a^b f(x)\,dx$.

Solution: If $f(x)$ is continuous on $[a, b]$ and $\Delta x = \frac{b-a}{n}$, then the definite integral is the limit of the Riemann sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

for all choices of $x_i^* \in [x_{i-1}, x_i]$.

II. Compute the derivatives of each of the following functions defined by integrals.

A. (5) $f(x) = \int_1^x e^{t^2}\,dt$

Solution: By the first part of the Fundamental Theorem of Calculus,

$$f'(x) = e^{x^2}.$$

B. (5) $g(x) = \int_{x^2}^4 \frac{\cos(t)}{t^2}\,dt$

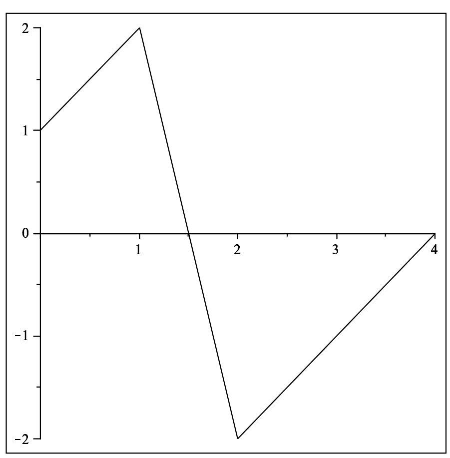
Solution: Reversing the limits of integration,

$$g(x) = -\int_4^{x^2} \frac{\cos(t)}{t^2}\,dt.$$

Then by the first part of the Fundamental Theorem and the Chain Rule,

$$g'(x) = -\frac{\cos(x^2)}{x^4} \cdot 2x = -\frac{2\cos(x^2)}{x^3}.$$

III. The following graph (made up of straight line segments) shows $y = f(t)$ for $0 \le t \le 4$.



Given: $f(1) = 2$, $f(2) = -2$, $f(3) = -1$, and $f(4) = 0$, The function F is defined by $F(x) = \int_0^x f(t)\,dt$.

A. (5) Determine the values $F(x)$ for $x = 0, 1, 2, 3, 4$ and enter them in the following table.

Solution: By computing areas of trapezoids, we have:

x	0	1	2	3	4
$F(x)$	0	3/2	3/2	0	-1/2

B. (5) Does $F(x)$ have any critical points? If so, say where. If not say why not.

Solution: By the first part of the Fundamental Theorem, $F'(x) = f(x) = 0$ when $x = 3/2$ and $x = 4$. (Since $x = 4$ is an endpoint of the interval, we will also accept just $x = 3/2$.)

C. (5) Over which interval(s) is $F(x)$ concave down?

Solution: $F(x)$ is concave down on intervals where $F''(x) < 0$, or $F'(x) = f(x)$ is *decreasing*. This occurs for $1 < x < 2$ here.

IV.

A. (5) Integrate with a suitable u-substitution: $\displaystyle\int_0^1 (4x^3+1)^{3/5}x^2\,dx$.

Solution: Note that the x^2 outside the power is du up to a constant for $u = 4x^3 + 1$. So do this substitution where $du = 12x^2\,dx$. So

$$\begin{aligned}\int_0^1 (4x^3+1)^{3/5}x^2\,dx &= \frac{1}{12}\int_{u=1}^{u=5} u^{3/5}\,du \\ &= \frac{5}{96}u^{8/5}\Big|_{u=1}^{u=5} \\ &= \frac{5}{96}\left(5^{8/5}-1\right)\end{aligned}$$

B. (5) Integrate with a suitable u-substitution: $\displaystyle\int \frac{x\sin(3x^2)}{\cos(3x^2)+1}\,dx$.

Solution: Let $u = \cos(3x^2) + 1$. Then $du = -\sin(3x^2)(6x)\ dx$ by the Chain Rule. So

$$\begin{aligned} \int \frac{x\sin(3x^2)}{\cos(3x^2)+1}\ dx &= -\frac{1}{6}\int \frac{du}{u} \\ &= -\frac{1}{6}\ln|u| + C \\ &= -\frac{1}{6}\ln|\cos(3x^2)+1| + C. \end{aligned}$$

C. (7.5) Integrate by parts: $\displaystyle\int x^2 \sin(5x)\ dx$

Solution: We let $u = x^2$, $dv = \sin(5x)\ dx$, so $du = 2x\ dx$ and $v = -\frac{1}{5}\cos(5x)$. Applying the parts formula,

$$\int x^2\sin(5x)\ dx = -\frac{x^2}{5}\cos(5x) + \frac{2}{5}\int x\cos(5x)\ dx.$$

Now do parts again on the remaining integral with $u = x$, $dv = \cos(5x)\ dx$, so $du = dx$ and $v = \frac{1}{5}\sin(5x)$. This gives:

$$\begin{aligned} \int x^2\sin(5x)\ dx &= -\frac{x^2}{5}\cos(5x) + \frac{2}{5}\left(\frac{x}{5}\sin(5x) - \frac{1}{5}\int \sin(5x)\ dx\right) \\ &= -\frac{x^2}{5}\cos(5x) + \frac{2x}{25}\sin(5x) + \frac{2}{125}\cos(5x) + C \end{aligned}$$

D. (7.5) Integrate with the partial fraction method: $\displaystyle\int \frac{3x^2+1}{x^2+5x+6}\ dx$

Solution: Since the degree of the top is equal to the degree of the bottom, we divide first: $3x^2 + 1 = 3(x^2 + 5x + 6) - 15x - 17$ so

$$\int \frac{3x^2+1}{x^2+5x+6}\ dx = \int 3 - \frac{15x+17}{x^2+5x+6}\ dx.$$

Now $x^2 + 5x + 6 = (x + 2)(x + 3)$ so the partial fractions are

$$\frac{15x+17}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}.$$

Clearing denominators,

$$15x + 17 = A(x + 3) + B(x + 2),$$

so with $x = -2$ we see $A = -13$ and with $x = -3$ we see $B = 28$. Then

$$\int 3 - \frac{15x+17}{x^2+5x+6}\ dx = \int 3 - \left(\frac{-13}{x+2} + \frac{28}{x+3}\right)\ dx = 3x + 13\ln|x+2| - 28\ln|x+3| + C.$$

E. (10) Integrate via trigonometric substitution: $\displaystyle\int \sqrt{36 - x^2}\ dx$

Solution: From the form of the integrand, we use $x = 6\sin\theta$, so $dx = 6\cos\theta\, d\theta$ and the integral becomes

$$\int \sqrt{36 - x^2}\; dx = \int \sqrt{36(1 - \sin^2\theta)}\, 6\cos\theta\, d\theta = 36 \int \cos^2\theta\, d\theta.$$

To integrate this, we use the half-angle formula:

$$36 \int \cos^2\theta\, d\theta = 36 \int \frac{1}{2}(1 + \cos(2\theta))\, d\theta = 18\theta + 9\sin(2\theta) + C$$

Now we convert back to expressions in the original variable x. From the substitution $x = 6\sin\theta$, we have $\theta = \sin^{-1}\left(\frac{x}{6}\right)$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\frac{x}{6}\frac{\sqrt{36-x^2}}{6} = \frac{x\sqrt{36-x^2}}{18}$. The integral equals

$$18\sin^{-1}\left(\frac{x}{6}\right) + \frac{x\sqrt{36 - x^2}}{2} + C.$$

(Note that you can check your work using #39 in the table!)

V. Integrate using any applicable method or the table. If you do use the table, give the number of the entry you are using.

A. (10) $\int \tan x \sec^4 x\; dx.$

Solution: Method 1: Since the derivative of $u = \sec x$ is $du = \sec x \tan x$, the form here is

$$\int \sec^3 x(\sec x \tan x)\, dx = \int u^3\; du = \frac{1}{4}u^4 + C = \frac{1}{4}\sec^4 x + C.$$

Method 2: Since the derivative of $\tan x$ is $\sec^2 x$, this one can also be done by splitting off two powers of $\sec x$ and converting the other $\sec^2 x$ to powers of $\tan x$:

$$\begin{aligned}
\int \tan x \sec^4 x\; dx &= \int \tan x(\sec^2 x)\sec^2 x\, dx \\
&= \int \tan x(1 + \tan^2 x)\sec^2 x\, dx \\
&= \int \tan x \sec^2 x\, dx + \int \tan^3 x \sec^2 x\, dx \\
&= \int u\, du + \int u^3\, du \qquad (\text{for } u = \tan x) \\
&= \frac{u^2}{2} + \frac{u^4}{4} + C \\
&= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C.
\end{aligned}$$

This is equivalent to the answer above via the trig identity $1 + \tan^2 x = \sec^2 x$.

Method 3: A third way to do this one is to convert everything to sines and cosines. The integral obtained that way can be done by a u-substitution letting $u = \cos x$, $du = -\sin x\, dx$:

$$\int \frac{\sin x}{\cos^5 x}\, dx = -\int u^{-5}\, du = \frac{u^{-4}}{4} + C = \frac{1}{4\cos^4 x} + C.$$

This is also equivalent to the answer in Method 1 above since $\sec x = \frac{1}{\cos x}$.

B. (10) $\displaystyle\int \frac{\sqrt{9+e^{2x}}}{e^x}\, dx$

Solution: Substitute $u = e^x$, then $du = e^x dx$, so $dx = \frac{du}{e^x} = \frac{du}{u}$. Making these substitutions yields

$$\int \frac{\sqrt{9+u^2}}{u^2}\, du,$$

which is # 24 in the table with $a = 3$. We apply the table entry:

$$= -\frac{\sqrt{9+u^2}}{u} + \ln\left|u + \sqrt{9+u^2}\right| + C = -\frac{\sqrt{9+e^{2x}}}{e^x} + \ln\left|e^x + \sqrt{9+e^{2x}}\right| + C.$$

Note that this is *not* # 23 because of the "extra" u in the denominator from dx.

C. (10) $\displaystyle\int \frac{dx}{x^2+8x+17}$.

Solution: Complete the square in the denominator: $x^2 + 8x + 17 = (x+4)^2 + 1$. Therefore, the substitution $u = x + 4$, $du = dx$ takes this to the basic form

$$\int \frac{1}{u^2+1}\, du = \tan^{-1}(u) + C = \tan^{-1}(x+4) + C.$$