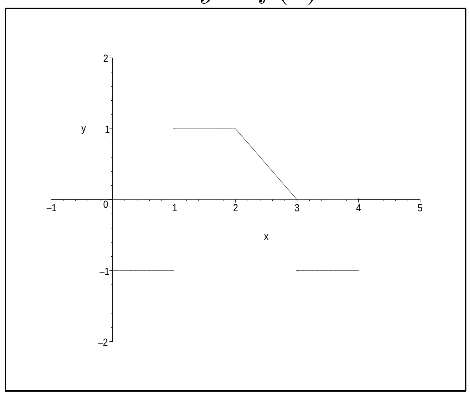
Holy Cross College, Spring Semester, 2005
MATH 132, Section 01, Final Exam Solutions
May 11

1. The following graph shows the function $y = f(x)$.



a. [10 points] Let $F(x)$ be the antiderivative of $f(x)$ with $F(0) = 0$. Compute the entries in the following table of values:
Solution:

x	0	1	2	3	4
$F(x)$	0	-1	0	.5	$-.5$

For example, the area between the graph and the x-axis from $x = 0$ to $x = 1$ (counted with a negative sign) is -1. So by the Fundamental Theorem of Calculus:

$$-1 = \int_0^1 f(x)\, dx = F(1) - F(0) = F(1) - 0 \Rightarrow F(1) = -1$$

The others are obtained similarly.

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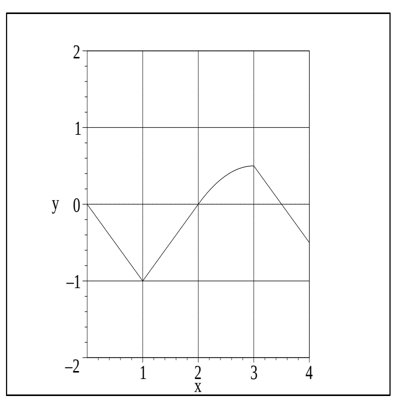
b. [10 points] Sketch the graph $y = F(x)$ on the interval $0 \le x \le 4$:



2. Integrate each of the following. You must show all work and cite any table entries you use for full credit.

a. [5 points] $\int \frac{x^2+3x+5}{x^{3/2}}\ dx$

Solution: Divide $x^{3/2}$ into each term on the top, then integrate:

$$\begin{aligned}\int \frac{x^2+3x+5}{x^{3/2}}\ dx &= \int x^{1/2}+3x^{-1/2}+5x^{-3/2} \\ &= \frac{2}{3}x^{3/2}+6x^{1/2}-10x^{-1/2}+C\end{aligned}$$

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b. [10 points] $\int \frac{e^t+1}{(e^t+t)^2}\ dt$

Solution: Let $u = e^t + t$, so $du = (e^t+1)\ dt$. Then the form is

$$\int \frac{du}{u^2} = \frac{-1}{u} + C = \frac{-1}{e^t+t} + C.$$

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c. [10 points] $\int \frac{dx}{(9-x^2)^{3/2}}$

Solution: We use the trigonometric substitution $x = 3\sin\theta$, so $dx = 3\cos\theta$ and

$$\begin{aligned}\int \frac{dx}{(9-x^2)^{3/2}} &= \int \frac{3\cos\theta\ d\theta}{27\cos^3\theta} \\ &= \frac{1}{9}\int \frac{d\theta}{\cos^2\theta} \\ &= \frac{1}{9}\tan\theta + C \\ &= \frac{1}{9}\frac{\sin\theta}{\cos\theta} + C \\ &= \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C\end{aligned}$$

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d. [10 points] $\int_0^1 x^2e^{2x}\ dx$

Solution: Use integration by parts twice, or # 14 in the table of integrals:

$$\int x^2e^{2x}\ dx = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)e^{2x} + C$$

Then for the definite integral

$$\int x^2 e^{2x}\ dx = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) e^{2x}|_0^1 = \frac{1}{4}(e^2 - 1) \doteq 1.597$$

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e. [10 points] $\int \frac{(x-1)\ dx}{x^3 + 25x}$

Solution: Use partial fractions based on the factorization $x^3 + 25x = x(x^2 + 25)$. We have

$$\frac{x-1}{x(x^2+25)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 25}.$$

Clearing denominators,

$$x - 1 = A(x^2 + 25) + (Bx + C)x = (A + B)x^2 + Cx + 25A$$

Equating coefficients of like powers of x: $(A + B) = 0$, $C = 1$, and $25A = -1$. So $A = \frac{-1}{25}$ and $B = \frac{-1}{25}$. Then using # 25 in the table

$$\int \frac{\frac{-1}{25}}{x} + \frac{\frac{1}{25}x + 1}{x^2 + 25} = \frac{-1}{25}\ln|x| + \frac{1}{50}\ln|x^2 + 25| + \frac{1}{5}\arctan\left(\frac{x}{5}\right) + C$$

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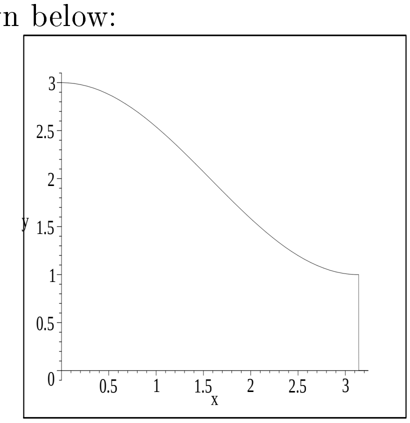
3. All parts of this problem refer to the region in the plane bounded by $y = 2 + \cos(x)$, $y = 0$, $x = 0$, and $x = \pi$, shown below:



a. [10 points] Find the volume of the solid with base R whose cross-sections by planes perpendicular to the x-axis are squares.

Solution: We have, using # 18 in the table

$$\begin{aligned} V &= \int_0^\pi (2+\cos x)^2 \ dx \\ &= \int_0^\pi 4 + 4\cos x + \cos^2 x \ dx \\ &= (4x + 4\sin x + \frac{1}{2}x + \frac{1}{2}\sin x \cos x|_0^\pi \\ &= \frac{9\pi}{2} \end{aligned}$$

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b. [10 points] True or False: The solid obtained by rotating R about the x-axis has volume π times the volume from part a. (For full credit, give the integral that would compute the volume of this solid of revolution, and answer the question.)

Solution: This is TRUE, since the volume of the solid of revolution is

$$\int_0^\pi \pi(2+\cos x)^2 \ dx = \pi \int_0^\pi (2+\cos x)^2 \ dx.$$

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c. [15 points] A thin metal plate has the shape of the region R (units in meters) and constant mass density $1kg/m^2$. Find the x-coordinate of its center of mass.

Solution: We have, integrating by parts on the top

$$\begin{aligned} \overline{x} &= \frac{\int_0^\pi x(2+\cos x) \ dx}{\int_0^\pi (2+\cos x) \ dx} \\ &= \frac{x^2 + x\sin x + \cos x|_0^\pi}{2x + \sin x|_0^\pi} \\ &= \frac{\pi^2 - 2}{\pi} \end{aligned}$$

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d. [10 points] Set up, *but do not evaluate* the integral giving the arclength of the top edge of R.

Solution: The arclength integral is $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ dx$. Here, since $y = 2 + \cos x$, $\frac{dy}{dx} = -\sin x$ and

$$L = \int_0^\pi \sqrt{1+\sin^2 x} \ dx.$$

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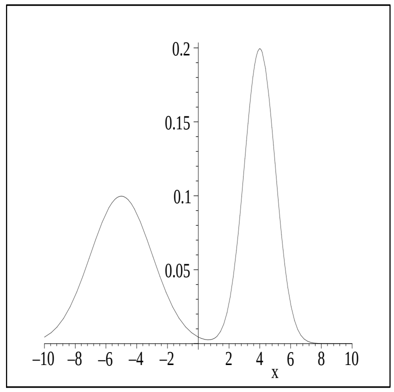
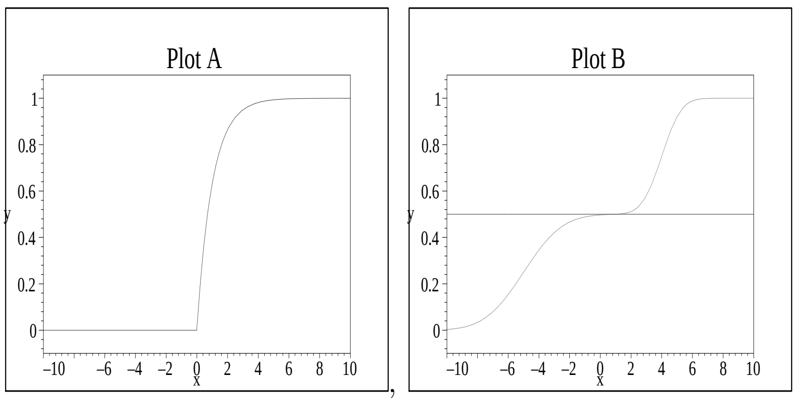
4. [5 points each] The following graph shows a probability density function (pdf) for a quantity x:



a. Which of the following two plots shows the corresponding cumulative distribution for x:



Solution: Graph B is the correct cdf for the given density function. We can see this because the given density should be the derivative of the cdf. Plot B has two maxima for the slope, corresponding to the two "peaks".

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b. Using the appropriate graph, estimate the *median* of x.

Solution: The median is the T-value where $\int_{-\infty}^{T} p(x)\ dx = \frac{1}{2}$. Drawing in a horizontal line in plot B, it appears that the line $y = 1/2$ intersects the graph at about $x = 1$. That is the median value.

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5. [5 points each] Does each of the following series converge? For full credit, you must justify your answer completely by showing how the indicated test leads to your stated conclusion.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ – alternating series test.

Solution: $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$ and for all $n \geq 1$, $\frac{1}{n^{1/3}} > \frac{1}{(n+1)^{1/3}}$. Hence the alternating series test (see p. 420 in the textbook) implies that this series *converges.*

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b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{9/10}}$ – integral test

Solution: The improper integral

$$\int_1^{\infty} \frac{dx}{x^{9/10}} = \lim_{b\to\infty} 10x^{1/10}|1^b$$

does not converge, because as $b \to \infty$, so does $b^{1/10}$. Hence by the integral test, this series *diverges.*

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c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^n}$ – any applicable method

Solution: Method 1: Since $n \geq 1$, $\frac{1}{ne^n} < \frac{1}{e^n}$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$ is a geometric series with ratio $\frac{1}{e} < 1$. Hence it converges. By the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^n}$ *converges too.*

Method 2: Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)e^{n+1}} \cdot \frac{ne^n}{1} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{e} = \frac{1}{e}$$

Since $\frac{1}{e} < 1$, the ratio test implies the series *converges.*

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6. [10 points] Using the ratio test, determine the radius of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2 4^n}$$

Solution:

$$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)^2 4^{n+1}} \cdot \frac{n^2 4^n}{(x+2)^n} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \frac{1}{4}|x+2| = \frac{1}{4}|x+2|$$

The radius of convergence is $1/(1/4) = 4$.

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7. All parts of this problem refer to $f(x) = \cos\left(\frac{x}{2}\right)$.

a. [15 points] Using the definition of Taylor polynomials, compute the Taylor polynomial of degree $n = 6$ for $f(x)$ at $a = 0$.

Solution: The problem said to use the definition of Taylor polynomials, so we compute $f'(x) = \frac{-1}{2} \sin\left(\frac{x}{2}\right)$, etc. The 6th degree polynomial is

$$p_6(x) = \sum_{i=0}^{6} \frac{f^{(i)}(0)}{i!} x^i = 1 - \frac{1}{8}x^2 + \frac{1}{384}x^4 - \frac{1}{46080}x^6$$

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b. [5 points] Use your answer from part a to approximate $\cos(1.2)$.

Solution: Since $f(x) = \cos\left(\frac{x}{2}\right)$, to approximate $\cos(1.2)$, we substitute $x = 2.4$ into $p_6(x)$ from part a. (Note: $p_6(x)$ is supposed to approximate $\cos\left(\frac{x}{2}\right)$, so to get 1.2 "inside" the cosine, we want $x = 2.4$. We get

$$p_6(2.4) = 1 - \frac{1}{8}(2.4)^2 + \frac{1}{384}(2.4)^4 - \frac{1}{46080}(2.4)^6 \doteq 1 - .72{.}0864 - .0041472 \doteq .3622528$$

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c. [10 points] Using the error bound for Taylor approximations, estimate how big n must be to guarantee 4 decimal place accuracy in approximations to $f(x)$ for all x between 0 and 2.

Solution: The $(n+1)$st derivative of $f(x)$ is $\frac{1}{2^{n+1}}$ times either $\pm \sin x$ or $\pm \cos x$. Hence we can use $M = \frac{1}{2^{n+1}}$ in the error bound, and we get for x between 0 and 2:

$$|f(x) - p_n(x)| \leq \frac{\frac{1}{2^{n+1}}}{(n+1)!} 2^{n+1} = \frac{1}{(n+1)!}.$$

We have $\frac{1}{8!} \doteq .000025$ so $n = 7$ (or $n = 6$, since $p_6 = p_7$ for cos) is large enough.

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8. When a course ends, it's an unfortunate fact of life that students start to forget the material they have learned. One mathematical model for this process states that *the rate at which a student forgets material is proportional to the difference between the material currently remembered and some positive constant* $a < 1$. Let y be the fraction of the original material remembered t weeks after the course has ended. Then $y(0) = 1$, and the model assumes $y(t) > a$ for all t.

a. [5 points] Which of the following differential equations is the translation of the model described above:

$$I : \frac{dy}{dt} = -k(y - a) \qquad II : \frac{dy}{dt} = \frac{-k}{y - a}$$

Solution: I is the correct equation.

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b. [10 points] Solve the equation you selected in part a by separation of variables.

Solution: Dividing by $y - a$ separates so

$$\begin{aligned} \int \frac{dy}{y-a} &= \int -k\ dt \\ \ln|y-a| &= -kt + c \\ y &= a + de^{-kt} \end{aligned}$$

where $d = \pm e^c$. Note that it is given in the problem that $y(0) = 1$. This says $1 = y(0) = a + d \cdot 1$, so $d = 1 - a$.

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c. [10 points] Suppose $a = .2$, and after 3 weeks the student only remembers 80% of the material from the course (that is, $y(3) = .8$). How long will it be before the student only remembers one half of the material?

Solution: Using this last comment, $y(t) = .2 + .8e^{-kt}$. From the given information $y(3) = .8$, so

$$.8 = .2 + .8e^{-3k} \Rightarrow k = \frac{\ln(.6/.8)}{-3} \doteq .0959$$

Then we want to solve for t:

$$.5 = .2 + .8e^{(-.0959)t} \Rightarrow t = \frac{\ln(.3/.8)}{-.0959} \doteq 10.23$$

That is, the student will only remember half of the material 10.23 weeks after the end of the class.