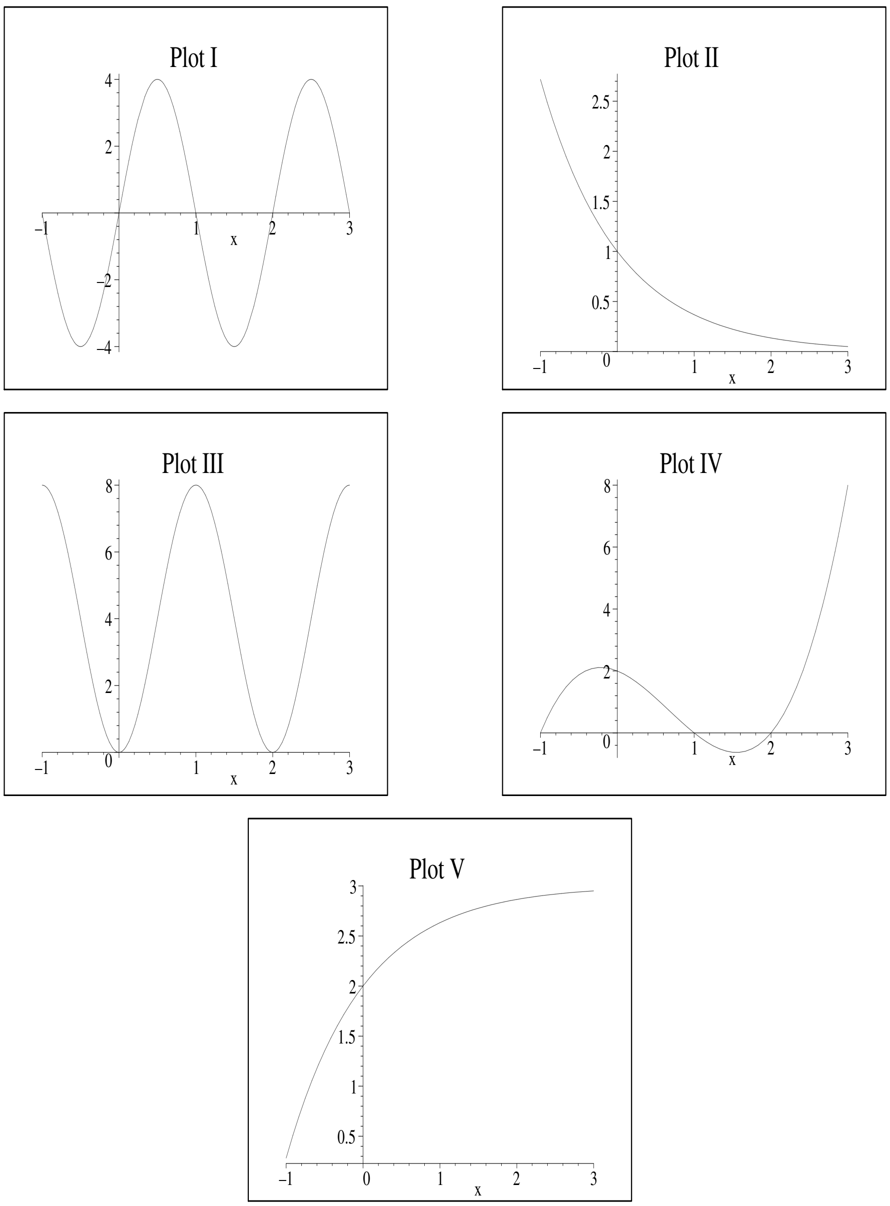
1. [5 points each] Circle the number of the graph showing each of the following functions.

- (a) $f(x) = 3 - e^{-x}$ *Answer:* V
- (b) $f(x) = x^3 - 2x^2 - x + 2$ *Answer:* IV
- (c) $f(x) = 4\sin(\pi x)$ *Answer:* I
- (d) $f(x) = 4 - 4\cos(\pi x)$ *Answer:* III
- Note: Graph II is $y = e^{-x}$. V is what you get from II, if that graph is reflected across x-axis and shifted up by 3.



2. [20 points] One of the functions given in the following table is linear and the other is exponential. Find formulas of the appropriate type for each.

x	1	2	3	4	5
$f(x)$	1.2	0.6	0.3	0.15	0.075
$g(x)$	-2.3	-0.6	1.1	2.8	4.5

Solution: $g(x)$ is linear, since the slopes of the lines between all pairs of points in the table is 1.7. For instance using the first two points: $m = \frac{-0.6+2.3}{2-1} = 1.7$. The equation for g is obtained by the point-slope form: $y - (-2.3) = 1.7(x-1)$, so $y = 1.7x - 4$.

Since the problem said one is exponential, that means that $f(x)$ is the exponential one. We can find the equation for $f(x) = ca^x$ as usual. From the table with $x = 1, 2$, $ca^1 = 1.2$ and $ca^2 = 0.6$ so $\frac{ca^2}{ca} = a = \frac{0.6}{1.2} = \frac{1}{2}$. Then from the first data point $c\frac{1}{2} = 1.2$, so $c = 2.4$. $g(x) = \frac{2.4}{2^x}$.

3.

- (a) [15 points] The depth of water in a tank oscillates sinusoidally once every 4 hours. The smallest depth is 2 feet and the maximum depth is 5 feet, which occurs at $t = 0$. Find a formula for the depth $d(t)$ if t is the time in hours.

  Solution: The amplitude is $\frac{5-2}{2} = \frac{3}{2}$. The vertical shift is $\frac{5+2}{2} = \frac{7}{2}$. Putting $t = 0$ at the start of the period where there is a maximum means we want to use cos. Finally the period is 4, so we get
$$d(t) = \frac{3}{2}\cos\left(\frac{\pi t}{2}\right) + \frac{7}{2}$$

- (b) [5 points] How fast is the depth changing at $t = 1.3$ hours? Is it increasing or decreasing?

  Solution: The question is asking for the derivative of $d(t)$ at $t = 1.3$: $d'(t) = -\frac{3\pi}{4}\sin\left(\frac{\pi t}{2}\right)$, so $d'(1.3) = -\frac{3\pi}{4}\sin\left(\frac{1.3\pi}{2}\right) \doteq -2.1$ feet per hour. Since this is negative, the depth is decreasing at $t = 1.3$.

4. Compute the following limits [5 points each]. Any legal method is OK.

- (a) *Solution:*

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+1}{x-3} &= \frac{\lim_{x\to 1} x^2+1}{\lim_{x\to 1} x-3}\\ &= \frac{2}{-2}\\ &= -1\end{aligned}$$

- (b) *Solution:* This limit is indeterminate of the form 0/0, so we use L'Hopital's Rule – twice!

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4x+4}{\cos(x-2)-1} &= \lim_{x\to 2}\frac{2x-4}{-\sin(x-2)}\\ &= \lim_{x\to 2}\frac{2}{-\cos(x-2)}\\ &= -2\end{aligned}$$

- (c) This one can also be done by L'Hopital, or by some algebraic "trickery":

$$\begin{aligned}\lim_{x\to\infty}\frac{5x^2-x+21}{3x^2+x+1} &= \lim_{x\to\infty}\frac{5x^2-x+21}{3x^2+x+1}\cdot\frac{1/x^2}{1/x^2}\\ &= \lim_{x\to\infty}\frac{5-1/x+21/x^2}{3+1/x+1/x^2}\\ &= \frac{5}{3}\end{aligned}$$

5.

- (a) [5 points] State the limit definition of the derivative. *Solution:*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(provided that this limit exists).

- (b) [10 points] Use the definition to compute $f'(x)$ for $f(x) = \sqrt{x}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

- (c) [10 points] Find the equation of the tangent line to the graph $y = \sqrt{x}$ at the point $[4, 2]$.

Solution: From part B, the derivative of $\sqrt{x}$ at $x = 4$ is $\frac{1}{2\sqrt{4}} = \frac{1}{4}$. This is the slope of the tangent line, which has the equation $y - 2 = \frac{1}{4}(x - 4)$, or $y = \frac{1}{4}x + 1$.

6. Compute the following derivatives using the derivative rules. You need not simplify. [5 points each]

- (a) $f(t) = t^3 - \frac{1}{\sqrt[4]{t}} + \pi^t$.

Solution: By the power and exponential rules,

$$f'(t) = 3t^2 + \frac{1}{4}x^{-5/4} + \pi^t \ln(\pi)$$

- (b) $g(x) = \dfrac{x^2 - 2}{\cos(x) + 1}$

Solution: By the quotient rule,

$$g'(x) = \frac{(\cos(x) + 1)(2x) - (x^2 - 2)(-\sin(x))}{(\cos(x) + 1)^2}$$

- (c) $h(z) = \ln(4z^2 + 2e^{\arctan(z)})$

Solution: By the chain rule,

$$h'(z) = \frac{8z + \frac{2e^{\arctan(z)}}{1+z^2}}{4z^2 + 2e^{\arctan(z)}}$$

- (d) Find $\frac{dy}{dx}$ if $x^2y - 2y^3 = 3$.

Solution: Differentiating implicitly, $x^2\frac{dy}{dx} + 2xy - 6y^2\frac{dy}{dx} = 0$. Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{-2xy}{-6y^2 + x^2}$$

7. Consider the family of curves defined by $y = f(x) = x^4 + 2ax^2$, where a is any fixed real number.

- (a) [10 points] Find the *critical points* of f, construct a sign diagram for $f'(x)$ in the case that $a < 0$. Which of your critical points are local maxima and which are local minima?

  Solution: We have $f'(x) = 4x^3 + 4ax = 4x(x^2 + a)$. When $a < 0$, this has three solutions, so three critical points: $x = -\sqrt{-a}, 0, \sqrt{-a}$. (Note: $a < 0$ means $-a > 0$, so $\sqrt{-a}$ does exist in the real numbers in this case.) The sign diagram should show $f' < 0$ for $x < -\sqrt{-a}$, $f' > 0$ for $-\sqrt{-a} < x < 0$, $f' < 0$ for $0 < x < \sqrt{-a}$ and $f' > 0$ for $x > \sqrt{-a}$. This means that $x = \pm\sqrt{-a}$ are local minima and $x = 0$ is a local maximum (First Derivative Test).

- (b) [10 points] Repeat part a, but assume now that $a > 0$.

  Solution: As before, $f'(x) = 4x^3 + 4ax = 4x(x^2 + a)$. But when $a > 0$, the equation $x^2 + a = 0$ has no real solutions. So there is only one critial point: $x = 0$. $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$. So $x = 0$ is a local minimum.

- (c) [10 points] How many different *inflection points* does the graph $y = f(x)$ have if $a < 0$? Explain.

  Solution: When $a < 0$, $f''(x) = 12x^2 + 4a = 0$ if $x = \pm\sqrt{-a/3}$. The sign of f'' changes at each, so there are *two* inflection points in this case.

---

8. [15 points] A cubical block of dry ice (solid CO_2) is evaporating and losing volume at the rate of 10 cm^3/min. How fast are the sides of cube shrinking when the block has volume 125 cm^3? Give the units of your answer.

Solution: Let x denote the side of the cube, so the volume is $V = x^3$. Differentiating with respect to t, we get $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$. We know $\frac{dV}{dt} = -10$ when $V = 125$. At that time, $x = (125)^{1/3} = 5$, so

$$\frac{dx}{dt} = \frac{-10}{3 \cdot 5^2} = \frac{-2}{15} \doteq -.13$$

(units: cm/min).

9. [20 points] Ship A travels along the path given by the parametric curve $x = t, y = t^2$. At the same time, ship B travels along the curve $x = t, y = 4t - 5$. At what time are the two ships *closest to one another*?

Solution: At each time t, we can find the distance between the ships using the distance formula for points in the plane:

$$d(t) = \sqrt{(x_A(t) - y_A(t))^2 + (y_A(t) - y_B(t))^2} = \sqrt{(t-t)^2 + (t^2 - 4t + 5)^2} = |t^2 - 4t + 5|$$

But note that $t^2 - 4t + 5 = (t-2)^2 + 1 > 0$ for all t. So, we get $d(t) = t^2 - 4t + 5$. We want the time when the ships are closest, so we are looking for the minimum of $d(t)$. To find this, take $d'(t) = 2t - 4$ and set $= 0$. This says $t = 2$.

Note: Many people made tables of the distance at whole number values for t: $t = 0, 1, 2, 3$, etc., and concluded that the minimum distance ocurred at $t = 2$ from that information. While the conclusion is correct, the reasoning is not. How do you know that the distance doesn't reach an even smaller value for some t between 1 and 2, or t between 2 and 3? So this solution did not receive full credit, even though the answer was correct.

10.

- (a) [7.5 points] Compute the left- and right-hand sums for the function $f(t) = e^{-t^2}$ on the interval $[0, 1]$, using $n = 4$ equal subdivisions.

  Solution: $\Delta t = \frac{1-0}{4} = .25$.

  $$LHS = e^0 \Delta t + e^{-(.25)^2} \Delta t + e^{-(.5)^2} \Delta t + e^{-(.75)^2} \Delta t \doteq .82$$

  and

  $$LHS = e^{-(.25)^2} \Delta t + e^{-(.5)^2} \Delta t + e^{-(.75)^2} \Delta t + e^{-1} \Delta t \doteq .66$$

  Note: A lot of people were computing values of this function incorrectly. I couldn't tell exactly what went wrong, but I think it was probably a calculator issue. So I didn't assess a large penalty if your method was OK, but the values were incorrect.

- (b) [7.5 points] Consider the graph $y = f(t)$ given below. Compute the average value $\overline{y}$ of f on the interval $0 \leq t \leq 5$, given that the area marked A is 25 square units, and the areas marked B and C are both 16 square units.

Solution: Since the areas A and C are below the t-axis, in computing $\int_0^5 f(t)\ dt$, we count them with negative signs. So the average value is

$$\overline{y} = \frac{1}{5-0}\int_0^5 f(t)\ dt = \frac{1}{5}(-25+16-16) = -5.$$