MATH 131, section 1 – Solutions for Practice Questions for Exam 2
October 17, 2007

1. All parts of this question refer to the parametric curve $x = 3\cos(2t)$, $y = 5\sin(2t)$.

a. Eliminate the parameter t and find a Cartesian equation for this curve.

Solution: We have $\frac{x}{3} = \cos(2t)$ and $\frac{y}{5} = \sin(2t)$, so

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2(2t) + \sin^2(2t) = 1.$$

The equation is $\frac{x^2}{9} + \frac{y^2}{25} = 1$, which we recognize as the equation of an *ellipse* centered at $(0,0)$, with semimajor axis 5 (along the y-axis), and semiminor axis 3 (along the x-axis).

b. What portion of the curve is traced out for $0 \le t \le \frac{\pi}{2}$, and in which direction is the curve being traced?

Solution: As t increases from 0 to $\frac{\pi}{2}$, $\cos(2t)$ is decreasing from 1 to -1, and $\sin(2t)$ is increasing from 0 to 1 at $t = \frac{\pi}{4}$, then decreasing back to 0 at $t = \frac{\pi}{2}$. Therefore, x is decreasing from 3 to -3 and y is increasing up to 5, then decreasing back to 0. The portion of the curve traced out is the top half of the ellipse and we are moving *counterclockwise.*

c. What would change in your answer to part b if the curve above was replaced by $x = 3\cos(-2t)$, $y = 5\sin(-2t)$?

Solution: We would be moving *clockwise* around the *bottom half* of the ellipse rather than counterclockwise around the top half.

2. Use the sum, product, and/or quotient rules to compute the following derivatives. You may use any correct method, but must show work and simplify your answers for full credit.

a. $\frac{d}{dx}\left(5x\sqrt{x} - \frac{2}{x^3} + 11x - 4\right)$

Solution: By the power and sum rules,

$$\frac{d}{dx}\left(5x\sqrt{x} - \frac{2}{x^3} + 11x - 4\right) = \frac{15}{2}x^{1/2} + 6x^{-4} + 11.$$

b. $\dfrac{d}{dt}(t^2e^t)$

Solution: By the product rule,

$$\frac{d}{dt}(t^2e^t) = t^2e^t + 2te^t = (t^2+2t)e^t.$$

c. $\dfrac{d}{dz}\dfrac{z^2-2z+4}{z^2+1}$

Solution: By the quotient rule,

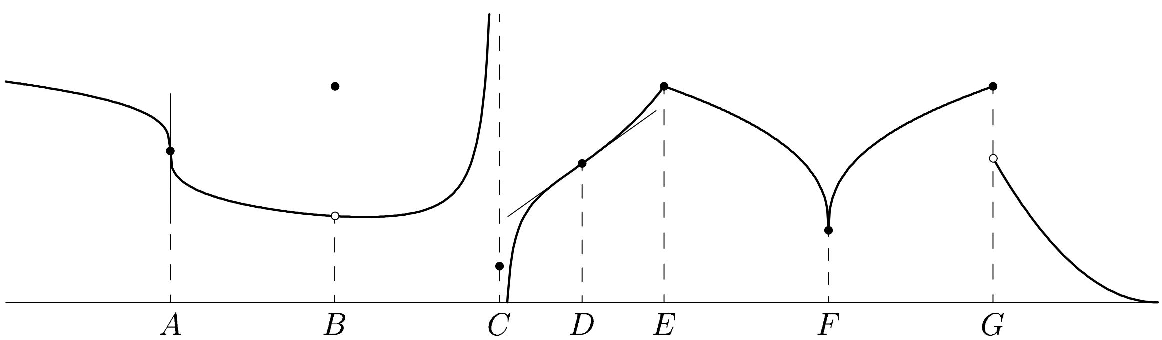
$$\begin{aligned}\frac{d}{dz}\frac{z^2-2z+4}{z^2+1} &= \frac{(z^2+1)(2z-2)-(z^2-2z+4)(2z)}{(z^2+1)^2} \\ &= \frac{2z^2-6z-2}{(z^2+1)^2}.\end{aligned}$$

d. $\dfrac{d}{dx}\left(\dfrac{\pi^2+\tan(e^\pi)-2x^e}{4}\right)$

Solution: Don't be fooled – most of this function is constant. The only part that is not is the $-\frac{x^e}{2}$, so the derivative is

$$\frac{d}{dx}\left(\frac{\pi^2+\tan(e^\pi)-2x^e}{4}\right) = \frac{d}{dx}\left(\frac{\pi^2}{4}+\frac{\tan(e^\pi)}{4}-\frac{x^e}{2}\right) = 0+0-\frac{ex^{e-1}}{2} = -\frac{ex^{e-1}}{2}.$$

3. The graph of a function f is shown below with several points marked. Find all the marked points at which the following are true, and give explanations for your answers.



a. f is discontinuous.

Solution: Points B, C, and G. Note that both one-sided limits exist and are equal at $x = B$, but the limit is not the same as the value of the function there (a removable discontinuity). At C, f has an infinite discontinuity (vertical asymptote). At G, f has a jump discontinuity (both one-sided limits exist but they are unequal).

b. f is continuous, but the graph of f has a vertical tangent line.

Solution: This happens at point A only.

c. f is continuous, but the graph of f has no tangent line.

Solution: This happens at points E (a "corner") and F (a cusp).

4. Compute the indicated limits. Show all work for full credit.

a. $\displaystyle\lim_{x\to 1} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4}$

Solution: We have

$$\lim_{x\to 1} 3x^2 - 5x - 2 = 3(\lim_{x\to 1} x)^2 - 5\lim_{x\to 1} x - 2 = 3 - 5 - 2 = -4$$

and

$$\lim_{x\to 1} x^2 - 4x + 4 = (\lim_{x\to 1} x)^2 - 4\lim_{x\to 1} x + 4 = 1 - 4 + 4 = 1$$

by the limit sum and product rules. Since the limit of the denominator is not zero, the limit quotient rule then says

$$\lim_{x\to 1} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4} = \frac{\lim_{x\to 1} 3x^2 - 5x - 2}{\lim_{x\to 1} x^2 - 4x + 4} = \frac{-4}{1} = -4.$$

b. $\displaystyle\lim_{x\to 2} \frac{3x^2 - 5x - 2}{x^2 - 4x + 4}$

Solution: We cannot apply the technique used in part a here since now

$$\lim_{x\to 2} 3x^2 - 5x - 2 = 3(\lim_{x\to 2} x)^2 - 5\lim_{x\to 2} x - 2 = 12 - 10 + 2 = 0.$$

and

$$\lim_{x\to 2} x^2 - 4x + 4 = (\lim_{x\to 2} x)^2 - 4\lim_{x\to 2} x + 4 = 4 - 8 + 4 = 0.$$

For $x \neq 2$, we have after factoring and cancellation:

$$\frac{3x^2 - 5x - 2}{x^2 - 4x + 4} = \frac{(3x+1)(x-2)}{(x-2)(x-2)} = \frac{3x+1}{x-2}.$$

Since the denominator is still going to 0 as $x \to 2$ while the numerator is approaching 7, this limit *does not exist.* In terms of the one-sided limits, in fact,

$$\lim_{x\to 2^-} \frac{3x+1}{x-2} = -\infty, \text{ while } \lim_{x\to 2^+} \frac{3x+1}{x-2} = +\infty.$$

c. $\displaystyle\lim_{x\to\infty} \frac{3x^2-5x-2}{x^2-4x+4}$

Solution: Divide the top and bottom by the fastest growing power of x that appears and then take the limit:

$$\begin{aligned}\lim_{x\to\infty} \frac{3x^2-5x-2}{x^2-4x+4} &= \lim_{x\to\infty} \frac{(3x^2-5x-2)\frac{1}{x^2}}{(x^2-4x+4)\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{3-\frac{5}{x}-\frac{2}{x^2}}{1-\frac{4}{x}+\frac{4}{x^2}} \\ &= 3/1 \\ &= 3.\end{aligned}$$

(Recall, this computation would show that the graph $y = \dfrac{3x^2-5x-2}{x^2-4x+4}$ has a horizontal asymptote at $y=3$ as $x \to +\infty$ and also as $x \to -\infty$.)

d. $\displaystyle\lim_{x\to 2^+} \frac{|x-2|}{x^2-5x+6}$

Solution: Note $x^2-5x+6 = (x-2)(x-3)$. Hence the function is

$$\frac{|x-2|}{(x-2)(x-3)} = \frac{|x-2|}{x-2}\cdot\frac{1}{x-3} = \begin{cases}\frac{1}{x-3} & \text{if } x>2 \\ \frac{-1}{x-3} & \text{if } x<2\end{cases}.$$

This shows

$$\lim_{x\to 2^+} \frac{|x-2|}{x^2-5x+6} = \lim_{x\to 2^+} \frac{1}{x-3} = -1.$$

5. Let $f(x) = x^3 - x^2$.

a. Find all intervals on which f is **decreasing**.

Solution: To answer this, we need $f'(x)$, since f is decreasing on an interval when $f'(x) < 0$ on that interval. Here $f'(x) = 3x^2 - 2x = x(3x-2)$. The graph $y = f'(x)$ is a parabola opening up. We see $f'(x) = 0$ at $x = 0$ and $x = 2/3$, so $f'(x)$ is *negative* for $0 < x < 2/3$ and positive for $x < 0$, $x > 2/3$. *Answer*: In interval notation, f is decreasing for $x \in (0, 2/3)$.

b. Find all intervals on which f is **concave up**.

Solution: To answer this, we need $f''(x)$, since f is concave up on an interval when $f''(x) > 0$ on that interval. We have $f''(x) = 6x - 2$, so $f''(x) < 0$ for $x < 1/3$ and $f''(x) > 0$ for $x > 1/3$. *Answer*: f is concave up for $x \in (1/3, +\infty)$

c. Find all intervals on which f is both **increasing** and **concave down**.

Solution: We want the intervals where $f'(x) > 0$ and $f''(x) < 0$. From the solutions for parts a, b, this is true only when $x < 0$. *Answer*: f is increasing and concave down for $x \in (-\infty, 0)$.

6. Do not use the differentiation rules from Chapter 3 in this question.

a. State the limit definition of the derivative $f'(x)$.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists.

b. Use the definition to compute the derivative function of $f(x) = \dfrac{1}{3x}$.

Solution: Using the definition from part a,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\dfrac{1}{3(x+h)} - \dfrac{1}{3x}}{h} \\ &= \lim_{h\to 0} \frac{3x - 3(x+h)}{9hx(x+h)} \quad \text{(simplify fraction)} \\ &= \lim_{h\to 0} \frac{-3h}{9hx(x+h)} \quad \text{(cancel like terms on top)} \\ &= \lim_{h\to 0} \frac{-1}{3x(x+h)} \quad \text{(cancel the } 3h \text{ top and bottom)} \\ &= \frac{-1}{3x^2}. \end{aligned}$$

c. Find the equation of the line tangent to the graph $y = \dfrac{1}{3x}$ at $x = 2$.

Solution: When $x = 2$, $f(2) = \dfrac{1}{6}$, and $f'(2) = \dfrac{-1}{12}$ from part b. So we calculate the point-slope form of the equation of the tangent line, which is $y - \dfrac{1}{6} = \dfrac{-1}{12}(x-2)$, or after simplifying,

$$y = \frac{-1}{12}x + \frac{1}{3}.$$

7. The total cost (in \$) of repaying a car loan at interest rate of $r\%$ per year is $C = f(r)$.

a. What is the meaning of the statement $f(7) = 20000$?

Solution: This means that if the interest rate is 7% per year, then the cost of repaying the loan is \$20000.

(a) What is the meaning of the statement $f'(7) = 3000$? What are the units of $f'(7)$?

Solution: This means that when $r = 7\%$ per year, the rate of change of the cost of repaying the loan is \$3000 dollars per % per year (the units are dollars per % per year).