Mathematics 244, section 1 – Linear Algebra
Solutions for Exam 2 – March 23, 2007

I. (15) Let V be a vector space and let S be a finite subset of V with $\text{Span}(S) = V$. If T is a linearly independent subset of V, show that $|T| \leq |S|$. (Here $|S|$ denotes the number of elements in S, and similarly for $|T|$.)

Solution: Say $S = \{v_1, \ldots, v_m\}$ and $T = \{w_1, \ldots, w_n\}$ in V, where $\text{Span}(S) = V$. We will show that if $n > m$, then T must be linearly dependent. Since $\text{Span}(S) = V$, for each j, $1 \leq j \leq n$, there exist scalars a_{ij} such that

$$w_j = a_{1j}v_1 + \cdots + a_{mj}v_m. \tag{1}$$

Suppose we have scalars $c_1, \ldots, c_n$ such that

$$0 = c_1w_1 + c_2w_2 + \cdots + c_nw_n. \tag{2}$$

Then substituting from the equations (1), we have

$$0 = c_1(a_{11}v_1 + \cdots + a_{m1}v_m) + c_2(a_{12}v_1 + \cdots + a_{m2}v_m) + \cdots + c_n(a_{1n}v_1 + \cdots + a_{mn}v_m).$$

Rearranging this equation gives:

$$0 = (a_{11}c_1 + \cdots + a_{1n}c_n)v_1 + \cdots + (a_{m1}c_1 + \cdots + a_{mn}c_n)v_n.$$

This will be satisfied if

$$\begin{aligned} a_{11}c_1 + \cdots + a_{1n}c_n &= 0 \\ \vdots \qquad\quad & \quad \vdots \\ a_{m1}c_1 + \cdots + a_{mn}c_n &= 0 \end{aligned}$$

But this is a homogeneous system of m linear equations in the variables $c_1, \ldots, c_n$. Since we assume that $n > m$, there are free variables, hence nontrivial solutions. This implies that T is *linearly dependent* if $n > m$ from (2). Hence if T is linearly independent, then $n = |T| \leq |S| = m$.

II. All parts of this question refer to the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 & s & 2 \\ -2 & -1 & -1 & 2s & -1 \\ 1 & 3 & 1 & s^2 & -1 \end{pmatrix}$$

A) (15) For which value(s) of the scalar s does A satisfy $\dim \text{Nul}(A) = 2$?

Solution: We begin by applying row operations $R_2 \mapsto R_2 + 2R_1$, $R_3 \mapsto R_3 - R_1$, and then $R_3 \mapsto R_3 + R_2$ the matrix A.

$$\begin{pmatrix} 1 & -2 & 0 & s & 2 \\ -2 & -1 & -1 & 2s & -1 \\ 1 & 3 & 1 & s^2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 & s & 2 \\ 0 & -5 & -1 & 4s & 3 \\ 0 & 5 & 1 & s^2 - s & -3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 0 & s & 2 \\ 0 & -5 & -1 & 4s & 3 \\ 0 & 0 & 0 & s^2 + 3s & 0 \end{pmatrix}$$

From this, we can see that there are always pivots in columns 1 and 2. Moreover, if $s^2 + 3s \neq 0$, then there is a pivot in column 4 as well. To get $\dim \mathrm{Nul}(A) = 2$ (not 3), we need three pivot columns (2 free variables), so the condition is $s^2 + 3s \neq 0$. $\dim \mathrm{Nul}(A) = 2$ for all real s other than $s = 0, s = -3$.

B) (10) Say $s = 1$. Give bases of $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$.

Solution: We substitute $s = 1$ in the last matrix above and continue the reduction to row-reduced echelon form. The result is:

$$\begin{pmatrix} 1 & 0 & 2/5 & 0 & 4/5 \\ 0 & 1 & 1/5 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus, a basis for $\mathrm{Col}(A)$ is given by columns 1,2,4 of the original matrix A (taking $s = 1$ of course):

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

A basis for $\mathrm{Nul}(A)$ comes from parametrizing the solutions of $Ax = 0$ with the free variables x_3, x_5 as usual:

$$\left\{ \begin{pmatrix} -2/5 \\ -1/5 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

III. (10) Use Cramer's Rule to solve the system

$$3x_1 + 2x_2 = 9$$
$$-8x_1 + 3x_2 = -12$$

Solution: We have

$$x_1 = \frac{\det \begin{pmatrix} 9 & 2 \\ -12 & 3 \end{pmatrix}}{\det \begin{pmatrix} 3 & 2 \\ -8 & 3 \end{pmatrix}} = 51/25$$

and

$$x_2 = \frac{\det\begin{pmatrix} 3 & 9 \\ -8 & -12 \end{pmatrix}}{\det\begin{pmatrix} 3 & 2 \\ -8 & 3 \end{pmatrix}} = 36/25$$

IV. Let $V = M_{2\times 2}(\mathbf{R})$, the vector space of all 2×2 matrices with real entries. Let $W = \mathbf{R}^2$.

A) (10) Is $T : V \to W$ defined by $T(A) = \begin{pmatrix} \det(A) \\ \det(A^2) \end{pmatrix}$ a linear mapping? Why or why not?

Solution: No, T is not a linear mapping. The easiest way to see this is to note that if we multiply the matrix A by a scalar c, then

$$T(cA) = \begin{pmatrix} \det(cA) \\ \det((cA)^2) \end{pmatrix} = \begin{pmatrix} c^2 \det(A) \\ c^4 \det(A^2) \end{pmatrix}.$$

If $c \neq 1$, then this is not the same as $cT(A)$. For instance if $A = I$ and $c = 2$, we get

$$T(2I) = \begin{pmatrix} 4 \\ 16 \end{pmatrix} \neq 2T(I) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

B) (10) Show that

$$H = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V : a + b = c + d \right\}$$

is a vector subspace of V.

Solution: The zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in H since $0 + 0 = 0 + 0$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are in H, then $a + b = c + d = 0$ and $e + f = g + h = 0$. Then $A + B = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$ and

$$(a + e) + (b + f) = (a + b) + (e + f) = (c + d) + (g + h) = (c + g) + (d + h)$$

(using commutativity and associativity of addition in $\mathbf{R}$). Hence $A + B \in H$. Finally, if $A \in H$ and $r \in \mathbf{R}$, then $rA = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$ and $ra+rb = r(a+b) = r(c+d) = rc+rd$. Therefore $rA \in H$.

C) (5 **Extra Credit**) Find a basis for H from part B and determine its dimension.

Solution: In the equation $a+b-c-d=0$, b, c, d are free variables. Hence $\dim H = 3$, and a basis consists of

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

V. (True-False) For each true statement, give a short proof. For each false statement, give a counterexample. (Note: As always, "true" means "true in every case." Partial credit *will* be given for definitions or theorems that apply even if you don't find a complete solution.)

A) (10) Let A be a 4×4 matrix in which $R_1 + 2R_2 = R_3 + R_4$ (R_i is row i of the matrix). Then $\det(A) = 0$.

Solution: This is *True*. For a proof, we will use the fact that "replacement operations" $R_i \mapsto R_i + cR_j$ do not change the value of the determinant. Hence if we apply the row operations $R_1 \mapsto R_1 + 2R_2$, and $R_4 \mapsto R_4 + R_3$, we will obtain matrix with two equal rows (rows 1 and 4). Hence if we apply another operation such as $R_4 \mapsto R_4 - R_1$, we obtain a row of zeroes. Expanding the determinant along row 4, we obtain $\det(A) = 0$.

B) (10) There exist linear mappings $T : \mathbf{R}^2 \to \mathbf{R}^2$ with standard matrices A satisfying $\mathrm{Nul}(A) = \mathrm{Col}(A)$.

Solution: This is *True*. Since the statement says "there exists such a matrix," it suffices to produce one. From the equation $\dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A) = 2$, we can see that to get equality, we want both terms $= 1$. This means that we want a 2×2 matrix A whose columns are scalar multiples, and such that $Ax = 0$ for $x =$ both of its columns. The condition $\dim \mathrm{Col}(A) = 1$ means that

$$A = \begin{pmatrix} a & ca \\ b & cb \end{pmatrix}$$

for some a, b, c. Then

$$\begin{pmatrix} a & ca \\ b & cb \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

implies that $a^2 + cab = 0$ and $ab + cb^2 = 0$. There are infinitely many different solutions of these equations. If we take $c = -1$, for instance, then $a = 1, b = 1$ gives one:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

is such a matrix, with

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \mathrm{Nul}(A).$$

(There are infinitely many others too.)

C) (10) Let A, B, C be $n \times n$ matrices. If $\det(A^t BC^5) = 23$, then A, B, C are *all* invertible matrices.

Solution: This is *True*. From properties of determinants we know

$$23 = \det(A^t BC^5) = \det(A)\det(B)(\det(C))^5.$$

Since the product is $23 \neq 0$, $\det(A), \det(B), \det(C)$ must all be $\neq 0$. This means that A, B, C are all invertible.