Mathematics 244, section 1 – Linear Algebra
Solutions for Exam 1 – February 16, 2007

If A, B are $n \times n$ matrices and A is invertible, then there is a unique $n \times n$ matrix X with $AXA = B$.

I.

A) (10) For which value(s) of $c \in \mathbf{R}$ is the system of linear equations represented by the following augmented matrix *consistent*?

$$\begin{pmatrix} -1 & 2 & 0 & c & | & 1 \\ 0 & 1 & 2 & 0 & | & -1 \\ 2 & -4 & 0 & 8 & | & 5 \end{pmatrix}$$

Solution : Perform the row operation $R_3 \mapsto R_3 + 2R_1$:

$$\begin{pmatrix} -1 & 2 & 0 & c & | & 1 \\ 0 & 1 & 2 & 0 & | & -1 \\ 0 & 0 & 0 & 8+2c & | & 7 \end{pmatrix}$$

If $8 + 2c = 0$, then the system is *inconsistent*, otherwise it is consistent. So the answer here is that the system is consistent for all $c \in \mathbf{R}$, $c \neq -4$.

B) (10) Give a parametrization of the set of solutions for a value of c (your choice) that makes the system consistent.

Solution : We will take $c = 0$ for simplicity. Continuing from the matrix in the solution for part A, we obtain the row-reduced echelon form

$$\begin{pmatrix} 1 & 0 & 4 & 0 & | & -3 \\ 0 & 1 & 2 & 0 & | & -1 \\ 0 & 0 & 0 & 1 & | & 7/8 \end{pmatrix}$$

This gives a system where x_3 is a free variable. Set of solutions is

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 7/8 \end{pmatrix} + t \begin{pmatrix} -4 \\ -2 \\ 1 \\ 0 \end{pmatrix} : t \in \mathbf{R} \right\}$$

II.

A) (5) *Define:* The set S of vectors in $\mathbf{R}^n$ is *linearly independent.*

S is linearly independent if whenever a linear combination

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

for $v_i \in S$, then $c_1 = c_2 = \cdots = c_k = 0$.

B) (10) For which value of $a \in \mathbf{R}$ is

$$S = \left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix} \right\}$$

a linearly *dependent* subset of $\mathbf{R}^3$?

Solution : We do row operations on the matrix with columns equal to the vectors in S:

$$\begin{pmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & a \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & a \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}$$
$$\mapsto \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -2a \\ 0 & -1 & 1-3a \end{pmatrix}$$
$$\mapsto \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -2a \\ 0 & 0 & 1-5a \end{pmatrix}$$

In order for the set to be linearly dependent, the matrix must have fewer than 3 pivot columns, which means $1 - 5a = 0$, so $a = 1/5$.

C) (15) Let $b = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$. Using your value of a from part B, is $b \in \mathrm{Span}(S)$? Why or why not?

Solution : We do the row operations as in the previous part, but on the augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -1 & 1 & 3 \\ 2 & 1 & 0 & 3 \\ 1 & 0 & 1/5 & 4 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 0 & 1/5 & 4 \\ 2 & 1 & 0 & 3 \\ 3 & -1 & 1 & 3 \end{array}\right)$$
$$\mapsto \left(\begin{array}{ccc|c} 1 & 0 & 1/5 & 4 \\ 0 & 1 & -2/5 & -5 \\ 0 & -1 & 2/5 & -9 \end{array}\right)$$
$$\mapsto \left(\begin{array}{ccc|c} 1 & 0 & 1/5 & 4 \\ 0 & 1 & -2/5 & -5 \\ 0 & 0 & 0 & -14 \end{array}\right)$$

This says the system is inconsistent, so b is *not* in the span of S.

III. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the mapping defined by

$$T(x) = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix} x$$

A) (10) Show that T is a linear mapping by verifying the properties in the definition.

Solution : For *any* 3×3 (or $m \times n$) matrix, $A(x+y) = Ax + Ay$ and $A(cx) = cAx$. This says $T(x+y) = T(x) + T(y)$ and $T(cx) = cT(x)$, so T is linear.

B) (10) Is T an onto mapping? Why or why not?

Solution : There are a number of ways to answer this. One of the most direct is just to see whether the matrix is *invertible*. If so, the Invertible Matrix Theorem says T is onto. We follow our procedure for computing the matrix inverse:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 6 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 6 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 6 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 & 2 & -1 & -2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -2 & 1 & 3 \end{pmatrix}$$

Since A is invertible the answer is yes – the mapping T is onto.

IV. (True-False) Determine whether each of the following statements is true or false. For the ones that are true, give short proofs; for those that are false, give counterexamples.

A) (10) If the homogeneous system $Ax = 0$ has two free variables, then the set of solutions of $Ax = b$ is a plane for all b.

Solution : This is FALSE because the system $Ax = b$ could also be inconsistent for some b. An example is given by

$$[A|b] = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & -1 \end{array}\right)$$

B) (10) If $x \in \text{Span}(S)$ (*and* $x \notin S$), then the set $S \cup \{x\}$ is linearly dependent.

Solution : This is TRUE. If $x = c_1 v_1 + \cdots + c_k v_k$, where $v_i \in S$ for all i, then

$$0 = (-1)x + c_1 v_1 + \cdots + c_k v_k.$$

Since the coefficient $-1 \neq 0$ in this linear combination, the set $S \cup \{x\}$ is linearly dependent.

Comment: Without the added statement $x \notin S$, this would actually be FALSE(!) Counterexample: $S = \{e_1, e_2\}$ in $\mathbf{R}^n$ for any $n \geq 2$. $e_1 = 1 \cdot e_1 + 0 \cdot e_2$ is in $\text{Span}(S)$, but $S \cup \{e_1\} = \{e_1, e_2\}$ is still linearly independent.

C) (10) Let A be an $m \times n$ matrix that defines an *onto* mapping $T_A : \mathbf{R}^n \to \mathbf{R}^m$ ($T_A(x) = Ax$ for $x \in \mathbf{R}^n$). Then the span of the set of columns of A is $\mathbf{R}^m$.

Solution : This is TRUE. Since T is onto, for every $y \in \mathbf{R}^m$, there is some $x \in \mathbf{R}^n$ such that $T(x) = Ax = y$. If $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then

$$y = Ax = x_1 a_1 + \cdots + x_n a_n$$

where $a_1, \ldots, a_n$ are the columns of A . This shows that every $y \in \mathbf{R}^m$ is a linear combination of the columns of A.