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# Some Remarks on Fitzpatrick and Flynn's Gröbner Basis Technique for Padé Approximation

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**Abstract**

In Fitzpatrick and Flynn (1992), a Gröbner basis technique for multivariable Padé approximation problems was developed under a rather restrictive hypothesis on the shape of the numerator and denominator in relation to the approximation conditions desired. In this article, we show that their hypotheses can be replaced by other less stringent conditions, and we show how to compute some standard forms of multivariable approximants through several examples.

## 1. Introduction

The general Padé approximation problem can be phrased as the question of finding rational functions $a/b$ in one or more variables of some specified form, that approximate some other given function $h$ of the same variables in a suitable sense. For instance, we might ask that $a/b$ interpolate values of $h$ (or minimize a measure of the interpolation error), or that specified initial segments of the Taylor expansions of $a/b$ and $h$ at a point agree. Padé-type approximations are used in both numerical and symbolic computation and have a number of applications in numerical analysis, in coding theory (in decoding algorithms for multidimensional cyclic codes, for instance), and in multidimensional signal processing (for instance, in design of IIR filters).

Fitzpatrick and Flynn (1992) introduced a symbolic technique for computing Padé approximants using the theory of Gröbner bases for submodules of free modules over polynomial rings (see Cox, Little, and O'Shea (1998), Adams and

Loustaunau (1994)). Given a polynomial $h \in k[x_1, \ldots, x_s]$ representing an initial segment of the Taylor series of the function to be approximated, and an ideal $I \subset k[x_1, \ldots, x_s]$ encoding the desired agreement conditions between the approximant and the polynomial $h$ at $x = 0$, they consider the module $M$ of solutions $(a, b) \in (k[x_1, \ldots, x_s])^2$ of the congruence

$$a \equiv bh \bmod I. \tag{1}$$

Each pair $(a, b)$ with $b(0) \neq 0$ yields a rational function $a/b$ that approximates $h$ modulo $I$. For example, in the case $s = 1$, if we seek an approximant $a(x)/b(x)$ with $\deg(a) \leq n$, $\deg(b) \leq m$, and $b(0) = 1$ then there are $n + m + 1$ free coefficients in $a(x)$ and $b(x)$. So one usually takes $I = \langle x^{n+m+1} \rangle$, expecting to be able to match a general $h(x)$ modulo $\langle x^{n+m+1} \rangle$ with $a(x)/b(x)$.

The basis of Fitzpatrick and Flynn's method is a statement guaranteeing that a particular solution of the Padé approximation problem appears in a suitable Gröbner basis for the module $M$ of solutions of (1). To describe this, we begin by fixing some monomial order $>$ on $k[x_1, \ldots, x_s]$. Given two monomials $\varphi_1$ and $\varphi_2$ in $k[x_1, \ldots, x_s]$, Fitzpatrick and Flynn introduce the so-called *weak term order condition* with respect to $(\varphi_1, \varphi_2)$. Let $(a, b)$ be a solution of (1) where $a, b$ are relatively prime, and both $a, b$ are reduced modulo $I$. Then $(a, b, I)$ satisfies the weak term order condition with respect to $(\varphi_1, \varphi_2)$ if $\varphi_1 \geq LT_>(a)$, $\varphi_2 \geq LT_>(b)$, and for all monomials $\rho$ and $\sigma$ in $k[x_1, \ldots, x_s]$ such that $\varphi_1 \geq \rho$, $\varphi_2 \geq \sigma$, and $\rho, \sigma \notin LT_>(I)$, then the product $\rho\sigma \notin LT_>(I)$. Then Fitzpatrick and Flynn's main result is the following statement.

Theorem 1.1 (Fitzpatrick and Flynn (1992), Theorem 2.4): *Let (1) have a solution $(a_0, b_0)$ where $a_0, b_0$ are relatively prime and reduced modulo $I$. Assume that $(a_0, b_0, I)$ satisfies the weak term order condition with respect to $(\varphi_1, \varphi_2)$. Then a constant multiple of $(a_0, b_0)$ appears in any Gröbner basis for $M = \{(a, b) | a \equiv bh \bmod I\}$ with respect to a "weighted" monomial order $>_{(\varphi_1,\varphi_2)}$ on $(k[x_1, \ldots, x_s])^2$ defined by $x^\alpha e_i >_{(\varphi_1,\varphi_2)} x^\beta e_j$ if $x^\alpha \varphi_i > x^\beta \varphi_j$ or if $x^\alpha \varphi_i = x^\beta \varphi_j$ and $i < j$.*

The reason behind this result is that the weak term order condition implies that $(a_0, b_0)$ is *minimal* in the module $M$ with respect to the $>_{(\varphi_1,\varphi_2)}$ order and this implies that $(a_0, b_0)$ must appear in the Gröbner basis with respect to that order. The monomial order $>_{(\varphi_1,\varphi_2)}$ is an example of the class of orders used by Schreyer to develop Gröbner basis methods for syzygy computations (see Cox, Little, and O'Shea (1998), Chapter 5, Theorem 3.3), and indeed solving (1) is closely related to computing a module of syzygies.

Unfortunately, in many situations where methods for computing Padé approximants, or equivalently solving congruences of the form (1), are potentially of interest, especially in the study of decoding algorithms for multidimensional cyclic codes, this result does not apply. The reason is that the weak term order condition on $(a, b, I)$ as above is far too restrictive. In fact we will see examples later where that condition does not hold for the desired solution $(a, b)$ of the congruence (1) for any pair $(\varphi_1, \varphi_2)$.

Instead of requiring the weak term order condition, we will look for $(a, b)$ of a specific form:

$$\tau(a) < \tau(b) \leq m, \tag{2}$$

where $\tau(p)$ is the total degree of the polynomial $p$. In many cases, we will be able to show that if a solution of the form (2) exists, even if that desired module element is not *minimal* with respect to the monomial order used for the Gröbner basis computation, then an element of that form must still appear in a suitable Gröbner basis. Hence Fitzpatrick and Flynn's basic approach can be extended to a wider range of problems of this form than is apparent at first from the result quoted above.

As Fitzpatrick and Flynn also remark, applying Buchberger's algorithm for Gröbner bases directly to find multivariable Padé approximants does not offer any clear computational advantages over the linear algebra techniques used more commonly. Hence our contribution must also be seen as giving further theoretical understanding of this problem rather than as providing a superior method for computations.

The present paper is organized as follows. In section 2, we will introduce some terminology and notation for multivariable Padé approximations and describe the class of problems that we will consider. Section 3 will be devoted to the proofs of several general results giving results parallel to Fitzpatrick and Flynn's theorem quoted above, but without the restrictive hypothesis that the weak term order condition holds. Finally, in section 4, we will present a series of explicit examples illustrating the results of section 3.

The first steps of the work reported here were made as part of an undergraduate research project at the 1999 Summer Institute in Mathematics for Undergraduates (SIMU) at the University of Puerto Rico, Humacao carried out by three of us (Ortiz, Ortiz-Rosado, and Ríos-Soto), under the direction of the remaining two of us (Pablo and Little). We would like to thank the organizers of SIMU for providing a stimulating and enjoyable atmosphere.

## 2. Terminology and Notation

As a general reference for the general multivariable Padé approximation problem and the work that has been done there by function theorists and numerical analysts, we will use the survey article Cuyt (2000). Most of the examples of Padé approximants that we will consider will fall into the general category of *equation lattice* approximants described there, although we will also present some results about the so-called *homogeneous* approximants. We begin by sketching the connections between the general equation lattice framework with the algebraic formulation used by Fitzpatrick and Flynn.

All of our Padé approximants will be quotients of polynomials in $k[x_1, \ldots, x_s]$.

We use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_s$ for the total degree of a monomial $x^\alpha$, and $\tau(f)$ for the total degree of a polynomial $f(x) = \sum_\alpha c_\alpha x^\alpha$:

$$\tau(f) = \max_\alpha \{|\alpha| : c_\alpha \neq 0\}. \tag{3}$$

To describe the "shape" of the desired approximant $a/b \in k(x_1, \ldots, x_s)$ using the equation lattice approach, we specify finite subsets $N, D \subset \mathbf{Z}^s_{\geq 0}$ giving the possible exponents in the numerator and the denominator, respectively. We also specify a third finite subset $E \subset \mathbf{Z}_{\geq 0}$ describing the approximation properties we want. Using multi-index notation, we have

$$a(x) = \sum_{\alpha \in N} c_\alpha x^\alpha, \tag{4}$$

$$b(x) = \sum_{\beta \in D} d_\beta x^\beta. \tag{5}$$

Then given $h(x)$, we seek to determine $a(x), b(x)$ with

$$a(x) - b(x)h(x) = \sum_{\gamma \in \mathbf{Z}^s_{\geq 0} \setminus E} a_\gamma x^\gamma. \tag{6}$$

If $b(0) \neq 0$, then it is easy to see that equation (6) is equivalent to the statement that the coefficient of $x^\delta$ in the (formal) Taylor expansion of $a(x)/b(x) - h(x)$ is zero for all $\delta \in E$. We will denote by

$$\text{Padé}_{N/D,E}(h) \tag{7}$$

the set of all pairs $(a, b)$ where $a, b$ have the forms given in (4), (5) above solving (6). Each element of $\text{Padé}_{N/D,E}(h)$ defines a rational function $a/b$. However, there is a subtlety. In some particular solutions, there may be cancellations of factors between the numerator and denominator, and if so, after cancellation to $\overline{a}/\overline{b}$, the pair $(\overline{a}, \overline{b})$ may not solve (6).

In order for a Padé approximation problem of this form to be *well-posed* for a general $h(x)$ (that is, in order for solutions to exist and satisfy useful properties) it is common in the multivariable Padé approximation literature to require that the following *exponent conditions* hold.

1. The sets $N, D, E$ should be chosen so that $|N| + |D| \geq |E| + 1$.
2. If $\gamma \in E$, and $\gamma = \delta + \varepsilon$ for $\delta, \varepsilon \in \mathbf{Z}^s_{\geq 0}$, then $\delta, \varepsilon \in E$.

The first exponent condition says that if (6) is written as a system of linear equations in the coefficients of $a(x)$ and $b(x)$, then there are more variables than equations, so we expect a solution to exist for general $h(x)$. (For some $h$, the system may be inconsistent, though, so it is possible for $\text{Padé}_{N/D,E}(h)$ to be empty.) For the approximant to be unique, the inequality here should be an equality.

Example 1: In the case $s = 1$, the common form for Padé approximants corresponds to $D = \{0, 1, \ldots, m\}$, $N = \{0, 1, \ldots, n\}$, and $E = \{0, 1, \ldots, m + n\}$ for some $m, n \in \mathbf{Z}_{\geq 0}$. Both exponent conditions are clearly satisfied for these $N, D, E$. Moreover, if $b(0)$ is normalized to 1, we get the stronger condition $|N| + |D| = |E| + 1$.

Example 2: We will devote most of our attention to the multivariable Padé approximants of "triangular" form studied by Karlsson and Wallin (see Karlsson and Wallin (1977),Cuyt (2000)). These approximants are defined by bounds on the total degrees of monomials appearing in the numerator and the denominator. The exponent sets for the numerators and denominators are as follows: $N = \{\alpha : |\alpha| \leq n\}$, $D = \{\beta : |\beta| \leq m\}$ for some $m, n \in \mathbf{Z}_{\geq 0}$. Here there is a considerable amount of freedom in choosing $E$ to satisfy the exponent conditions. As we will see in Section 4, different choices yield approximants with different properties. Our desired solutions will always have $b(0) \neq 0$ in these examples.

Example 3: Another type of multivariable Padé approximant that has received much attention are the so-called *homogeneous* approximants (see Cuyt (2000)). These can be defined within the equation lattice framework by taking, for some $m, n \in \mathbf{Z}_{\geq 0}$: $N = \{\alpha : nm \leq |\alpha| \leq nm + n\}$, $D = \{\beta : nm \leq |\beta| \leq nm + m\}$, and $E = \{\gamma : nm \leq |\gamma| \leq nm + n + m\}$. Here the first exponent condition holds (with equality if $s = 2$), but the second does not. Also, $b(0) = 0$ in all cases by the form of $D$. In Section 4 we will see that homogeneous approximants can be obtained by replacing $N, D, E$ above by $N' = \{\alpha : 0 \leq |\alpha| \leq nm + n\}$, $D' = \{\beta : 0 \leq |\beta| \leq nm+m\}$, $E' = \{\gamma : 0 \leq |\gamma| \leq nm+n+m\}$. This choice does satisfy the second exponent condition. We will compute the desired element in $\text{Padé}_{N/D,E}(h)$ by finding an element in $\text{Padé}_{N'/D',E'}(h)$ in which the coefficients of monomials with $0 \leq |\gamma| \leq nm - 1$ in $a$ and $b$ all vanish.

The second exponent condition has an interesting interpretation and more far-reaching consequences. The following fact is straightforward and the proof will be left to the reader as an exercise.

Proposition 1: *Let $E \subset \mathbf{Z}^s_{\geq 0}$ satisfy the second exponent condition. Then the $k$-linear span of $\{x^\gamma : \gamma \in \mathbf{Z}^s_{\geq 0} \setminus E\}$ is a monomial ideal $I \subset k[x_1, \ldots, x_s]$.*

Hence, if the second exponent condition holds, then solving the equation lattice Padé problem (6) is the same as solving a congruence of Fitzpatrick and Flynn's form (1). The second exponent condition also implies the following covariance result.

Proposition 2 (Cuyt (2000), Theorem 3.1): *Let $(a, b) \in \text{Padé}_{N/D,E}(h)$ and let $h = 1/g$ with $g(0) \neq 0$. Then $(b, a) \in \text{Padé}_{D/N,E}(g)$.*

We note that because of Proposition 2, in a sense, it is enough to consider approximants $a/b$ with $LT_>(a) < LT_>(b)$. (If $LT_>(a) > LT_>(b)$, then we can use Proposition 2 to consider the "reciprocal" Padé problem for $b/a$ instead. If $LT_>(a) = LT_>(b)$ are equal, then writing $a = LT_>(a) + a'$ where $a'$ has only terms strictly smaller than $LT_>(a)$, and similarly for $b$,

$$\begin{aligned}\frac{a}{b} &= \frac{LT_>(a) + a'}{LT_>(a) + b'}\\ &= 1 + \frac{a' - b'}{LT_>(a) + b'}\end{aligned}$$

and then the problem reduces to finding an approximant for $h-1$ and note that now $LT_>(a'-b') < LT_>(a)$.) We will consider approximants satisfying a slightly different condition in the next section.

## 3. Theoretical Results

In this section, we will formulate and prove several theorems showing that, under the assumption that an approximant (that is, a solution of (1)) of a particular form exists, then an element of that form must appear in a Gröbner basis for the module of solutions with respect to a particular monomial order.

The particular form we consider is this: We require that in $(a_0, b_0)$, $\tau(b) \le m$ (where $m$ comes from the definition of $D$) and that $\tau(a_0) < \tau(b_0)$. Our motivation for focusing on this form comes primarily from a projected application to decoding problems for multidimensional cyclic codes. In our first result, we assume a uniqueness statement about the desired solution of the congruence.

For the Gröbner basis calculations, we need to introduce a monomial order on $(k[x_1, \ldots, x_s])^2$ that will "find" elements in the module $M$ of the desired form. Instead of Fitzpatrick and Flynn's weighted orders, we will consider orders $>_\tau$ defined as follows: Given two monomials $x^\alpha e_i, x^\beta e_j \in (k[x_1, \ldots, x_s])^2$, then $x^\alpha >_\tau x^\beta$ if:

1. $|\alpha| > |\beta|$, or if
2. $|\alpha| = |\beta|$ and $i < j$, or if
3. $|\alpha| = |\beta|$, $i = j$, and $x^\alpha > x^\beta$

for a fixed monomial order $>$ on $k[x_1, \ldots, x_s]$. (Note that for terms $x^\alpha e_i$ with $i$ fixed, the $>_\tau$ order is equivalent a graded order on $k[x_1, \ldots, x_s]$.)

THEOREM 3.1: *Assume that $N, D, E$ for an equation lattice Padé problem satisfy both exponent conditions, and let $I$ be the monomial ideal given by Proposition 1. Let*

$$M = \{(a, b) : a \equiv bh \bmod I\}$$

*and assume that, up to a constant multiple, there exists a unique $(a_0, b_0) \in M$ such that $\tau(a_0) < \tau(b_0) \le m$. Then a constant multiple of $(a_0, b_0)$ appears in any Gröbner basis for $M$ with respect to the $>_\tau$ order.*

**Proof:** The hypotheses on $(a_0, b_0)$ imply that $LT_{>_\tau}(a_0, b_0) = LT_>(b_0)e_2$. Let $\mathcal{G}$ be the Gröbner basis for $M$ with respect to the $>_\tau$ order. By the definition of a Gröbner basis, there exists $(r, s) \in \mathcal{G}$ such that $LT_{>_\tau} > (r, s)$ divides $LT_{>_\tau}(a_0, b_0)$. This shows that $LT_{>_\tau}(r, s) = LT_>(s)e_2$ and $LT_>(s)$ divides $LT_>(b_0)$. Suppose that $(r, s)$ is not a constant multiple of $(a_0, b_0)$. Then by the uniqueness hypothesis $\tau(s) > m$, or $\tau(r) \geq \tau(s)$. In the first case we get an immediate contradiction: we cannot have $\tau(b_0) \leq m$, $\tau(s) > m$, and $LT_>(s)|LT_>(b_0)$. If $\tau(r) \geq \tau(s)$, then by the definition of the $>_\tau$ order, $LT_{>_\tau}(r, s) = LT_>(r)e_1$. But this also leads to a contradiction, since then $LT_{>_\tau}(r, s)$ cannot divide $LT_{>_\tau}(a_0, b_0)$. The contradiction shows that $(r, s)$ must be a constant multiple of $(a_0, b_0)$. □

This hypothesis is still extremely restrictive. Nevertheless, it would apply in any Karlsson-Wallin Padé problem with denominator degree $m$, numerator degree $n < m$, $E$ chosen so that a unique solution was expected, and a sufficiently generic $h$. This would be true even if Fitzpatrick and Flynn's weak term order condition did not apply for any monomials $\varphi_1, \varphi_2$. We will see an example of this kind in Section 4.

Even if we do not require uniqueness of the approximant of the desired form, we can still show that some element of that form exists in a suitable Gröbner basis.

THEOREM 3.2: *Assume that $N, D, E$ for an equation lattice Padé problem satisfy both exponent conditions, let $I$ be the monomial ideal given by Proposition 1, and let*

$$M = \{(a, b) : a \equiv bh \bmod I\}.$$

*If there exists some $(a_0, b_0)$ in $M$ with $\tau(a_0) < \tau(b_0) \leq m$, then there exists some element $(r, s)$ in any Gröbner basis for $M$ with respect to the $>_\tau$ order such that $\tau(r) < \tau(s) \leq m$.*

**Proof:** Our hypotheses on $(a_0, b_0)$ imply that its $>_\tau$-leading term is $LT_>(b_0)e_2$. There must be some element $(r, s)$ in the $>_\tau$ Gröbner basis $\mathcal{G}$ for $M$ whose leading term divides $LT_>(b_0)e_2$. But then the leading term of $(r, s)$ must be $LT_>(s)e_2$, so $\tau(r) < \tau(s)$ by the definition of the $>_\tau$ order, and $LT_>(s)|LT_>(b_0)$ implies that $\tau(s) \leq m$. □

As a corollary of this theorem, we note the following fact which will be useful in the consideration of the homogeneous Padé approximants in Section 4.

COROLLARY 1: *In the situation of Theorem 3, let*

$$m_0 = \min\{\tau(b) : (a, b) \in M \text{ and } \tau(a) < \tau(b)\},$$

*and define*

$$M_0 = \{(a, b) \in M : \tau(b) = m_0 \text{ and } \tau(a) < \tau(b)\}.$$

*Assume that the minimal nonzero element of the module $M$ under the $>_\tau$ order lies in $M_0$. If $\mathcal{G}$ is a Gröbner basis for $M$ with respect to the $>_\tau$ order, then $\mathcal{G} \cap M_0$ spans $M_0$ over $k$.*

**Proof:** Apply the module version of the division, or normal form, algorithm (see Cox, Little, and O'Shea (1998), Chapter 5, Theorem 2.5) with divisors $\mathcal{G}$ and the $>_\tau$ monomial order. The remainder on division of each $(a, b) \in M_0 \subset M$, must be zero, so there are $(r_i, s_i) \in \mathcal{G}$ and polynomials $p_i$ such that

$$(a, b) = \sum_i p_i(r_i, s_i)$$

However, since $LT_{>_\tau}(p_i(r_i, s_i))$ is less than or equal to $LT_{>_\tau}(a, b)$ for all $i$, our hypotheses imply that each $(r_i, s_i) \in M_0$, and $p_i$ is constant for all $i$. $\square$

## 4. Examples

In this section, we will present several examples illustrating how the theorems from Section 3 can be applied to different types of Padé approximation problems. For all of the following computations, we used the implementation of Buchberger's algorithm in the `Groebner` package in Maple V Release 5 with $k = \mathbf{Q}$. The appropriate $>_\tau$ orders may all be defined by means of weight matrices.

We remark that these computations could also be done by the module version of the FGLM Gröbner basis conversion algorithm presented in Fitzpatrick (1997).

Example 4: We take $s = 2$ and consider equation lattice approximants of the shape specified by

$$\begin{aligned} N &= \{\alpha \in \mathbf{Z}^2_{\geq 0} : |\alpha| \leq 2\}, \\ D &= \{\beta \in \mathbf{Z}^2_{\geq 0} : |\beta| \leq 3\}, \\ I &= \langle x^4, x^3y^3, y^4 \rangle. \end{aligned}$$

That is, $E$ is the set of exponents of the monomials in the complement of the ideal $I$. This is a special case of the Karlsson-Wallin approximants considered in Section 2. Both of the exponent conditions are satisfied in this case. Indeed, $|N| + |D| = |E| + 1$, so we expect a unique approximant $a/b$ with $b(0) = 1$ in $\text{Padé}_{N/D,E}(h)$, at least for sufficiently general $h$.

We will consider first the Padé approximation problem for

$$\begin{aligned} h(x, y) &= 1 + 2\,y + 2\,y^2 + 4/3\,y^3 - x - 2\,xy - 2\,xy^2 - 4/3\,xy^3 + 1/2\,x^2 \\ &\quad + x^2y + x^2y^2 + 2/3\,y^3x^2 - 1/6\,x^3 - 1/3\,x^3y - 1/3\,x^3y^2 - 2/9\,x^3y^3 \end{aligned}$$

In this case, it can be verified by linear algebra techniques that there is a unique approximant in $\text{Padé}_{N/D,E}(h)$ and that the condition (2) is satisfied. Following

Fitzpatrick and Flynn's general approach, we compute a Gröbner basis for the module

$$M = \langle (h, 1), (x^4, 0), (x^3y^3, 0), (y^4, 0)\rangle$$

with respect to the $>_\tau$ order defined above using the lexicographic order $>$ with $x > y$ on $\mathbf{Q}[x, y]$. We find that the minimal element of the Gröbner basis gives our desired solution:

$$\begin{aligned}(a_0, b_0) \;=\;& (60 - 24\,x + 48\,y + 3\,x^2 - 12\,xy + 12\,y^2, 12\,xy^2 \\ & +36\,y^2 - 8\,y^3 + 36\,x - 72\,y + 60 - 36\,xy + 9\,x^2 - 6\,x^2y + x^3),\end{aligned}$$

As always, one could normalize to make the constant term in the denominator equal 1, and there is unique rational function corresponding to $(a_0, b_0)$ in $\text{Padé}_{N/D,E}(h)$.

We remark that Fitzpatrick and Flynn's weak termorder condition is not satisfied for any choice of $(\varphi_1, \varphi_2)$ in this example. Since $LT_>(a_0) = x^2$ and $LT_>(b_0) = x^3$ are not in $I$, but their product is in $I$, the weak term order condition cannot hold. However, even though Fitzpatrick and Flynn's theorem does not apply, our Theorem 2 does apply.

Example 5: Our next example uses the same $N, D, E$ as above, but a different, sparser, $h(x, y)$:

$$h(x, y) = -4\,x^2y^2 - 21\,xy^2 - 75\,y^2 + 93\,y^3 + 23\,y^4.$$

In this case, the set $\text{Padé}_{N/D,E}(h)$ is infinite (the equations (6) are underdetermined) but elements $(a, b) \in \text{Padé}_{N/D,E}(h)$ satisfying (2) with $b(0) \neq 0$ do exist. By our Theorem 3, we expect an element somewhere in the Gröbner basis that has the desired form. In the Gröbner basis for $\langle (h, 1), (x^4, 0), (x^3y^3, 0), (y^4, 0)\rangle$ with respect to the $>_\tau$ order, we find the minimal element $(a, b) = (0, y^2)$ which satisfies the congruence $a \equiv bh \bmod I$, but that is not of the desired form because $b(0) = 0$. The next largest element in the $>_\tau$ order also satisfies (2):

$$(-3515625y^2, 371\,x^3 + 7471\,x^2y + 58125\,y + 1175\,x^2 - 32550\,xy - 13125\,x + 46875)$$

and the corresponding $a_0/b_0$ is an approximant of the desired form. For the same reason as in the previous example, Fitzpatrick and Flynn's weak term order condition will not hold for any choice of $(\varphi_1, \varphi_2)$ here. Our Theorem 3 does apply, however.

Example 6: We retain $N$ and $D$ as above. In the previous two examples, we chose $I = \langle x^4, x^3y^3, y^4\rangle$ because the first exponent condition $|N| + |D| = |E| + 1$ was easily verified. In order to obtain Padé approximants with other useful properties, we may need to tailor the set $E$ for the problem at hand. For instance, one common requirement for multivariable Padé approximants in numerical analysis and other applications is that $a/b$ should satisfy the so-called *projection property*

– namely that if we set all but one variable equal to zero in the approximant (and if necessary cancel common factors in the numerator and denominator), the result should agree with the approximant for a suitable one-variable Padé problem where the function to be approximated is $h$ with all but that one variable set equal to zero. In Karlsson and Wallin (1977), a general condition implying the projection property is developed.

For simplicity, we only state this in the case $s = 2$. With a numerator of degree $n$ and a denominator of degree $m$, the one-variable approximants should agree with $h(x,0)$ and $h(0,y)$ $\bmod\langle x^{n+m+1}\rangle$ and $\langle y^{m+n+1}\rangle$, respectively. The projection property is valid for $(a,b) \in \text{Padé}_{N/D,E}(h)$, $N$ and $D$ as before if $E$ contains $\{(0,0),(1,0),\dots,(n+m,0),(0,1),\dots,(0,n+m)\}$. The simplest choice of $E$ for which this condition (with $n = 2$ and $m = 3$) and the second exponent condition both hold yields the monomial ideal $I = \langle x^6, x^3y, xy^3, y^6\rangle$. (Once again, it is easy to see that for general $h$, the weak term order condition will never be satisfied.)

We will take

$$h = -3 + xy^3 - 2\,x + y - 9\,x^2 - 7\,xy + x^4.$$

It is easy to check that using $N$ and $D$ as above but $I = \langle x^4, x^3y^3, y^4\rangle$ as in the previous examples, we get a unique element of $\text{Padé}_{N/D,E}(h)$, but that that approximant does not satisfy the projection property (we get $a(x,0) \equiv b(x,0)h(x,0) \bmod \langle x^4\rangle$, but not $\bmod \langle x^6\rangle$ as we should if the projection property were satisfied).

With $I = \langle x^6, x^3y, xy^3, y^6\rangle$, even though the equations for the approximant are underdetermined, the $>_\tau$ Gröbner basis for the module $M$ contains the element

$$\begin{aligned}(a_0, b_0) \;\; &= \;\; (125975\,xy + 34075\,x - 483\,y^2 + 164124\,x^2 + 53016 - 16223\,y\\ &\quad\;\; 47\,x^3 + 1029\,y^2x - 1974\,x^2 - 294\,xy + 423\,x - 483\,y - 17672)\end{aligned}$$

satisfying (2). Substituting $x = 0$ and $y = 0$ yields

$$\begin{aligned} a_0(x,0)/b_0(x,0) &= \frac{-3 - \frac{725}{376}\,x - \frac{873}{94}\,x^2}{1 - \frac{9}{376}\,x + \frac{21}{188}\,x^2 - \frac{1}{376}\,x^3},\\ a_0(0,y)/b_0(0,y) &= -3 + y.\end{aligned}$$

These agree with the one-variable Padé approximants for $h(x,0)$ and $h(0,y)$ with numerator degree $\le 2$ and denominator degree $\le 3$.

Example 7: Our final example will demonstrate how the homogeneous approximants described in Section 2 might be found in the present Gröbner basis framework, using Corollary 1. We consider

$$h = \sum_{k=0}^{5}\left(\frac{1}{k!}\sum_{i+j=k} x^i y^j\right).$$

(This is a Taylor polynomial for the function $f(x,y) = \frac{xe^y - ye^x}{x-y}$.) We want to determine a homogenous Padé approximant with $n = 1, m = 2$ (so the numerator will contain terms of total degree 2 and 3 and the denominator will contain terms of total degrees 2, 3, and 4).

Let $N = \{\alpha : |\alpha| \leq 3\}$, $D = \{\beta : |\beta| \leq 4\}$, and $I = \langle x, y\rangle^6$. The corresponding $E = \{\gamma : |\gamma| \leq 5\}$ has $|E| + 1 > |N| + |D|$, so the equations for Karlsson-Wallin type approximants are underdetermined. And in fact, computing a $>_\tau$ Gröbner basis for $M$ contains four elements of the form $(a, b)$ with $\tau(a) \leq 3$ and $\tau(b) \leq 4$, one of which is the minimal element of the Gröbner basis. It follows by Corollary 1 that the homogeneous approximant we are looking for corresponds to a linear combination of these four elements of the $>_\tau$ Gröbner basis, and indeed we find that the suitable linear combination (eliminating all $1, x, y$ terms) yields:

$$\begin{aligned}(a_0, b_0) \;=\; & (75\,y^2 + 75\,x^2 + 175\,xy^2 + 225\,xy + 175\,x^2y + 25\,y^3 + 25\,x^3, \\ & \frac{25}{2}\,x^4 - 125\,x^2y + 25\,xy^3 - 125\,xy^2 - 50\,y^3 + 25\,x^3y + 225\,xy \\ & + \frac{25}{2}\,y^4 + 75\,x^2 - 50\,x^3 + \frac{125}{2}\,x^2y^2 + 75\,y^2).\end{aligned}$$

The rational function $a_0/b_0$ is the desired homogeneous approximant.

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