Mathematics 134 – Intensive Calculus for Science 2
Practice Final Exam Solutions

I. A) The statement of the Fundamental Theorem of Calculus is: If f is continuous on $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

B) The function F is decreasing for $-2 < x < -1$ (since f is < 0 there), increasing for $-1 < x < 1$ (since f is > 0 on that interval, constant for $2 < x < 3$, and decreasing for $3 < x < 4$. For the intervals where f is horizontal, the slope of F is constant (so that part of the graph of F is a part of a straight line). The values of F at the points $-2, -1, 0, 1, 2, 3, 4$ can be found from the Fundamental Theorem. Since $F(0) = 0$ is given, by considering areas we see:

$$\begin{aligned}
F(0) - F(-2) &= \int_{-2}^0 f(x)\ dx = 0 \text{ so } F(-2) = 0 \\
F(0) - F(-1) &= \int_{-2}^0 f(x)\ dx = 1 \text{ so } F(-1) = -1 \\
F(1) - F(0) &= \int_0^1 f(x)\ dx = 1/2 \text{ so } F(1) = 1/2 \\
F(2) - F(0) &= \int_0^2 f(x)\ dx = 1/2 \text{ so } F(2) = 1/2 \\
F(3) - F(0) &= \int_0^3 f(x)\ dx = 3/2 \text{ so } F(3) = 3/2 \\
F(4) - F(0) &= \int_0^4 f(x)\ dx = 1/2 \text{ so } F(4) = 1/2
\end{aligned}$$

Here is a sketch of the plot:

II.
A) By the power (and sum and constant multiple) rules

$$\int 5x^4 - \sqrt{x} + \frac{3}{x}\ dx = x^5 - \frac{2}{3}x^{3/2} + 3\ln|x| + c$$

B) Let $u = x^4 + 1$, so $du = 4x^3 + 1$. The given integral is

$$\begin{aligned}\int x^3 \sin(x^4+1)\ dx &= \frac{1}{4}\int \sin(u)\ du \\ &= \frac{-1}{4}\cos(u) + c \\ &= \frac{-1}{4}\cos(x^4+1) + c\end{aligned}$$

C) Integrate by parts twice, letting u be the power of x and $dv = e^{-5x}dx$:

$$\begin{aligned}\int x^2 e^{-5x}\ dx &= \frac{-x^2}{5}e^{-5x} + \frac{2}{5}\int xe^{-5x}\ dx \\ &= \frac{-x^2}{5}e^{-5x} + \frac{2}{5}\left(\frac{-x}{5}e^{-5x} + \frac{1}{5}\int e^{-5x}\ dx\right) \\ &= \frac{-x^2}{5}e^{-5x} + \frac{2}{5}\left(\frac{-x}{5}e^{-5x} - \frac{1}{25}e^{-5x}\right) + c \\ &= \frac{-x^2}{5}e^{-5x} - \frac{2x}{25}e^{-5x} - \frac{2}{125}e^{-5x} + c\end{aligned}$$

(This can also be derived from entry 14 in the table of integrals, taking $a = -5$.)

D) Use entry 24 in the table, with $a = 3$, $u = 2x$. We have

$$\int \frac{1}{u^2+a^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + c$$

so

$$\begin{aligned}\int \frac{1}{(2x)^2+3^2}\ dx &= \frac{1}{2}\int \frac{1}{u^2+3^2}\ du \\ &= \frac{1}{2}\cdot\frac{1}{3}\arctan\left(\frac{x}{3}\right) + c \\ &= \frac{1}{6}\arctan\left(\frac{x}{3}\right) + c\end{aligned}$$

E) Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$, so the given integral is

$$\begin{aligned}\int \frac{(\ln(x))^4}{x}\ dx &= \int u^4\ du \\ &= \frac{u^5}{5} + c \\ &= \frac{(\ln(x))^5}{5} + c\end{aligned}$$

III. A) Rotating about the x-axis, we get disk cross-sections, so the volume is

$$V = \int_0^4 \pi(x^{3/2})^2\ dx = \int_0^4 \pi x^3\ dx = \pi\frac{x^4}{4}|_0^4 = 64\pi$$

B) Rotating about the line $y = -4$, we get "washer" cross-sections with inner radius 4 and outer radius $4 + x^{3/2}$. The volume is

$$\begin{aligned} V &= \int_0^4 \pi(x^{3/2}+4)^2 - \pi 4^2 \; dx \\ &= \pi \int_0^4 x^3 + 8x^{3/2} + 16 - 16 \; dx \\ &= \pi \left(\frac{x^4}{4} + \frac{16}{5} x^{5/2} \right) |_0^4 \\ &= \frac{832\pi}{5} \doteq 522.76 \end{aligned}$$

IV. A) Since the density is 0 for all $x < 0$, we have

$$\int_{-\infty}^{\infty} p(x)\; dx = \int_0^{\infty} .1e^{-.1x}\; dx = \lim_{b->\infty} -e^{-.1x}|_0^b = \lim_{b->\infty} (1 - e^{-.1b}) = 1$$

B) The median time between cars is the solution T of the equation $\int_{-\infty}^{T} p(x) = 1/2$. As in part A, since $p(x) = 0$ if $x < 0$, this says

$$1/2 = \int_0^T .1e^{-.1x}\; dx = 1 - e^{-.1T}.$$

Hence $e^{-.1T} = 1/2$, so $T = \ln(1/2)/(-.1) \doteq 6.93$.

V. A) To compute the derivatives here, you need to recall the product rule from last semester! $f(x) = xe^x$, so $f'(x) = xe^x + e^x$, $f''(x) = xe^x + e^x + e^x = xe^x + 2e^x$, and $f''(x) = xe^x + e^x + e^x + e^x = xe^x + 3e^x$. Therefore the 3rd degree Taylor polynomial at $a = 0$ is

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f''(0)}{6}x^3 = x + (2/2)x^2 + (3/6)x^3 = x + x^2 + x^3/2.$$

B) Both $p_3(.1)$ and $(.1)e^{.1}$ equal .1105 to four decimal places, so the agreement is close.

VI. A) From the Taylor series for $\frac{1}{1-u}$ at $a = 0$, we have $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$. Letting $u = -x^2$, we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

B) We know $\int \frac{1}{1+x^2}\; dx = \arctan(x)$. Hence to get the Taylor series for arctan we can integrate series for $\frac{1}{1+x^2}$ term by term. The result is

$$\arctan(x) = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for some constant c. Since $\arctan(0) = 0$, c must be zero, and

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

C) The series for $g(x) = \frac{\arctan(x)-x}{x^3}$ can be found by algebraic manipulations of the series from part B:

$$\frac{\arctan(x) - x}{x^3} = \frac{x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots - x}{x^3} = \frac{-1}{3} + \frac{1}{5}x^2 + \cdots$$

The coefficient of x^2 is the second derivative of $g(x)$ at zero, divided by 2, so $g''(0) = \frac{2}{5} > 0$. Hence g is concave up at 0.

VII. A) The number of calories needed per day to maintain weight is $20W$, so if I is the constant calorie intake per day,

$$\frac{dW}{dt} = k(20W - I)$$

for some constant of proportionality k.

B) Separating variables and integrating:

$$\int \frac{dW}{20W - I} = \int k\ dt$$

$$\frac{1}{20} \ln|20W - I| = kt + c$$

$$20W - I = de^{20kt}$$

$$W = \frac{I + de^{20kt}}{20}$$

C) Given $I = 3000$, and $k = \frac{-1}{3500}$. When $t = 0$, $W(0) = 160$, so we have $160 = \frac{3000+d}{20}$, so $d = 3200 - 3000 = 200$. Then the person's weight at the end of one year is

$$W(365) = \frac{3000 + 200e^{-365/3500}}{20} \doteq 151.2$$

Extra Credit: The idea here is to integrate the equations of the parabolas to find the areas of the given cross-sections, then use those to derive a right-hand sum approximation of the integral $\int_0^{20} A(x)\ dx$ with $\Delta x = 5$ to approximate the volume enclosed by the hull. For example, the area of the cross-section at $x = 5$ is $\int_{-1}^{1} 0 - (2x^2 - 2)\ dx = 8/3$. Doing the same for the other given cross-sections, the volume is estimated by

$$(8/3) \cdot 5 + 4 \cdot 5 + (16/3) \cdot 5 + \frac{32}{3\sqrt{5}} \cdot 5 \doteq 83.9\text{ft}^3$$