Mathematics 134 – Intensive Calculus for Science 2
Final Exam Solutions
May 7, 2002

I. A) The time interval $\Delta t = 10$ minutes, or $1/6$ hours. Since the speeds are given in miles per hour, we will use $\Delta t = 1/6$. The left-hand sum approximation to the total distance traveled is

$$LHS = (35 + 36 + 71 + 23 + 27 + 40) \doteq 38.7 \text{ miles}$$

The right-hand sum approximation is

$$RHS = (36 + 71 + 23 + 27 + 40 + 35) \doteq 38.7 \text{ miles}$$

also. Our best estimate is the average, 38.7 miles.

B) This was a "trick question". Ordinarily, we would use the relation $|RHS - LHS| = |v(1) - v(0)|\Delta t$ (*Note: mentioning this fact earned full credit*) to answer a question like this. However, since the velocity is not steadily increasing or decreasing, it is not possible to say that the actual distance traveled lies between the LHS and RHS values. *It is not possible to determine the information asked for from the information given.*

II. A) $\int_0^9 f(x)\, dx =$ area above x-axis, minus area below x-axis. Using the area formulas for circles and trapezoids (or triangles and rectangles), we get

$$\int_0^9 f(x)\, dx = 4 + \frac{1}{4}\pi(3)^2 - \frac{1}{4}\pi(4)^4 = 4 - \frac{7\pi}{4} \doteq -1.5$$

(Note: the integral only goes from $x = 0$ to $x = 9$. The triangle below the x-axis from $x = 9$ to $x = 12$ should be ignored.)

B) Let f be continuous on the interval $[a, b]$, and let F be any antiderivative of f. Then $\int_a^b f(x)\, dx = F(b) - F(a)$.

III. A) Use the power rule. Note: $\frac{5}{\pi}$ is a constant!

$$\int x^7 - 3x^{-1/2} + \frac{5}{\pi}\, dx = \frac{x^8}{8} - 6x^{1/2} + \frac{5}{\pi}x + C$$

B) Use substitution: $w = x^2 + 1$, $dw = 2x\, dx$. Then

$$\int xe^{x^2+1}\, dx = \frac{1}{2}\int e^w\, dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{x^2+1} + C$$

C) Use integration by parts, twice, with $u =$ power of x (or the table)

$$\begin{aligned}
\int x^2 \cos(3x)\ dx &= \frac{x^2}{3}\sin(3x) - \frac{2}{3}\int x\sin(3x)\ dx \\
&= \frac{x^2}{3}\sin(3x) - \frac{2}{3}\left(\frac{-x}{3}\cos(3x) + \frac{1}{3}\int \cos(3x)\ dx\right) \\
&= \frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C
\end{aligned}$$

D) This has the form of # 28 in the table with $x \to 2x$, $a = 1$:

$$\int \frac{1}{\sqrt{1-4x^2}}\ dx = \frac{1}{2}\int \frac{2}{\sqrt{1-(2x)^2}}\ dx = \frac{1}{2}\arcsin(2x) + C$$

E) This one is slightly tricky. The most direct way to do it is to add and subtract a term on the top to break the fraction up into two things that are easy to integrate. The second integral is du/u:

$$\begin{aligned}
\int \frac{1}{1+e^x}\ dx &= \int \frac{1+e^x-e^x}{1+e^x}\ dx \\
&= \int \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\ dx \\
&= \int 1\ dx - \int \frac{e^x}{1+e^x}\ dx \\
&= x - \ln(1+e^x) + C
\end{aligned}$$

(Another way is to use a substitution $u = e^x$. After some simplification, the form is # 26 in the table with $a = -1$, $b = 0$.)

IV. A) Rotating about the x-axis, the cross-sections of the solid are disks, radius $4 - x^2$, thickness dx. So the volume is

$$V = \int_0^1 \pi(4-x^2)^2\ dx = \pi\int_0^1 16 - 8x^2 + x^4\ dx = \pi(16x - \frac{8x^3}{3} + \frac{x^5}{5})|_0^1 = \frac{203\pi}{15}$$

B) Subdivide the plate into strips with long edges parallel to the y-axis. If $\Delta x =$ width is small enough, then the density is nearly constant on each strip and the mass of the strip is approximately (density)(area):

$$\begin{aligned}
\text{TotalMass} &\doteq \sum(1+x)(4-x^2)\Delta x \\
&= \int_0^1 (1+x)(4-x^2)\ dx \\
&= \int_0^1 4 + 4x - x^2 - x^3\ dx \\
&= (4x + 2x^2 - \frac{x^3}{3} - \frac{x^4}{4})|_0^1 = \frac{65}{12}
\end{aligned}$$

C) The x-coordinate of the center of mass is

$$\overline{x} = \frac{\int_0^1 x(1+x)(4-x^2)\ dx}{\int_0^1 (1+x)(4-x^2)\ dx}$$

V. A) The probability is

$$\int_1^2 \frac{2}{9}x(3-x)\ dx = \frac{2}{9}\left(\frac{3x^2}{2} - \frac{x^3}{3}\right)|_1^2 = \frac{13}{27}$$

B) The mean snowfall is

$$\int_0^3 x \cdot \frac{2}{9}x(3-x)\ dx = \frac{3}{2}$$

VI. With $f(x) = x\cos(3x)$, we differentiate using the product and chain rules, and simplify as we go along:

$$\begin{aligned}
f(x) &= x\cos(3x) & f(0) &= 0 \\
f'(x) &= \cos(3x) - 3x\sin(3x) & f'(0) &= 1 \\
f''(x) &= -6\sin(3x) - 9x\cos(3x) & f''(0) &= 0 \\
f'''(x) &= -27\cos(3x) + 27x\sin(3x) & f'''(0) &= -27 \\
f''''(x) &= 108\sin(3x) + 81x\cos(3x) & f''''(0) &= 0 \\
f^{(5)}(x) &= 405\cos(3x) - 243x\cos(3x) & f^{(5)}(0) &= 405
\end{aligned}$$

So

$$p_5(x) = x - \frac{27}{3!}x^3 + \frac{405}{5!}x^5 = x - \frac{9}{2}x^3 + \frac{27}{8}x^5$$

B) Using the Taylor series for $\cos(x)$:

$$x\cos(3x) = x\left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots\right) = x - \frac{9}{2}x^3 + \frac{27}{8}x^5 + \cdots$$

This agrees with part A.

VII. Let H be the horseshoe temperature. The surrounding constant temperature is the water temperature 30. Newton's Law says:

$$\frac{dH}{dt} = k(H - 30)$$

for some constant of proportionality k. So

$$\begin{aligned}\int \frac{dH}{H-30} &= \int k \; dt \\ \ln|H-30| &= kt + c \\ H - 30 &= be^{kt} \qquad (b = \pm e^c) \\ H &= 30 + be^{kt}\end{aligned}$$

We are given $H(0) = 100$, so $b = 70$. When $t = 10$, $T(10) = 70$, so

$$70 = 30 + 70e^{10k} \Rightarrow k = \frac{\ln(4/7)}{10} \doteq -.005596$$

Then we must solve

$$35 = 30 + 70e^{-.005596t} \Rightarrow t = \frac{\ln(5/70)}{-.005596} \doteq 47$$

(47 minutes).