Mathematics 134 – Intensive Calculus for Science 2
Solutions – Exam 3
April 29, 2002

I.

A) In order for $p(x)$ to be a pdf, we must have

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} p(x)\ dx \\ &= \int_0^1 c(x^2 - x^3)\ dx \\ &= c\left(\frac{x^3}{3} - \frac{x^4}{4}\right|_0^1 \\ &= c \cdot \frac{1}{12}\end{aligned}$$

So $c = 12$.

B) The mean recovery time is

$$\begin{aligned}\int_{-\infty}^{\infty} xp(x)\ dx &= \int_0^1 12x(x^2 - x^3)\ dx \\ &= 12\int_0^1 x^3 - x^4\ dx \\ &= 12\left(\frac{x^4}{4} - \frac{x^5}{5}\right|_0^1 \\ &= \frac{12}{20} = \frac{3}{5}\end{aligned}$$

hours (or 36 minutes).

II. This is a geometric series with $a = 1 + b$ and common ratio $x = 1 + b$, so from the summation formula

$$2 = \frac{1+b}{1-(1+b)} = \frac{1+b}{-b}$$

solving for b, we get $-2b = 1 + b$, so $3b = -1$, or $b = -1/3$.

III.

A) By the derivative chain rule, we have

$$\begin{aligned}f(x) &= \sin(2x + \pi/2) \Rightarrow f(0) = \sin(\pi/2) = 1 \\ f'(x) &= 2\cos(2x + \pi/2) \Rightarrow f'(0) = 2\cos(\pi/2) = 0 \\ f''(x) &= -4\sin(2x + \pi/2) \Rightarrow f''(0) = -4\sin(\pi/2) = -4 \\ f'''(x) &= -8\cos(2x + \pi/2) \Rightarrow f'''(0) = -8\cos(\pi/2) = 0 \\ f''''(x) &= 16\sin(2x + \pi/2) \Rightarrow f''''(0) = 16\sin(\pi/2) = 16\end{aligned}$$

Hence the fourth-degree Taylor polynomial is

$$p_4(x) = 1 - \frac{4}{2!}x^2 + \frac{16}{4!}x^4 = 1 - 2x^2 + \frac{2}{3}x^4$$

B) We know the Taylor series for $\cos(x)$ is

$$\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots$$

substituting $u = 2x$ we get

$$\cos(2x) = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} + \cdots = 1 - 2x^2 + \frac{2}{3}x^4 + \cdots$$

C) The answers are the same (because of the trig identity $\sin(\theta + \pi/2) = \cos(\theta)$, and the fact that the odd-degree terms in the cosine series are zero. The first three nonzero terms give the fourth degree Taylor polynomial).

IV.

A) For $y = xe^x$, the product rule gives

$$\frac{dy}{dx} = e^x + xe^x \tag{1}$$

The right side is

$$\frac{y(1+x)}{x} = \frac{xe^x(1+x)}{x} = e^x(1+x) = e^x + xe^x \tag{2}$$

Since (1) and (2) are the same function for all x, y satisfies the differential equation.

B) The correct slope field is 1. You can tell because the slope function $(y-1)(y-5)$ must be zero along the two horizontal lines $y = 1$ and $y = 5$. Only 1 does that.

C) The solution of 1 with $y(0) = 3$ decreases toward the line $y = 1$ (horizontal asymptote). The solution of 2 with $y(0) = 3$ starts off with a horizontal tangent, but then increases without any bound.