Mathematics 133 – Intensive Calculus for Science 1
Solutions – Exam 3
December 3, 2001

I.

A) Using the derivative rule for ln and the chain rule:

$$f'(x) = \frac{1}{x} + 2^{4x}\ln(2)\cdot 4 = \frac{1}{x} + 4\ln(2)2^{4x}$$

B) Using the product and chain rules and the derivative rules for sin and cos:

$$g'(x) = \sin(x)\cdot - \sin(x^2)\cdot 2x + \cos(x^2)\cdot cos(x) = -2x\sin(x)\sin(x^2) + \cos(x)\cos(x^2)$$

C) Using the derivative rule for arctan and the chain rule:

$$h'(x) = \frac{1}{1+(x^2+e^{2x}+1)^2}\cdot(2x+2e^{2x})$$

II.

A) Using the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^2+4)\cdot 1 - x\cdot 2x}{(x^2+4)^2} \\ &= \frac{-x^2+4}{(x^2+4)^2} \\ &= \frac{(2-x)(2+x)}{(x^2+4)^2} \end{aligned}$$

(factoring using the pattern for a difference of squares).

B) From the factored form of the numerator, the critical points (x where $f'(x) = 0$) are $x = \pm 2$. The denominator of $f'(x)$ is always positive, so the sign of f' is determined by the numerator. $-x^2 + 4$ is > 0 if $-2 < x < 2$ and positive when $x > 2$ or $x < -2$. The First Derivative Test shows that $x = -2$ is a *local minimum*, while $x = 2$ is a *local maximum.*

C) Differentiating f' with the quotient rule again, we get

$$f''(x) = \frac{(x^2+4)^2\cdot(-2x) - (-x^2+4)\cdot 2(x^2+4)\cdot 2x}{(x^2+4)^4}$$

We want to simplify this. *Note that there are several common factors in the terms on the top of this expression. WE DO NOT WANT TO MULTIPLY THIS OUT!!* Instead, we factor out $2x(x^2+4)$, cancel with the denominator, and simplify to obtain:

$$\begin{aligned} f''(x) &= \frac{2x(x^2+4)(-(x^2+4) - 2(-x^2+4))}{(x^2+4)^4} \\ &= \frac{2x(x^2-12)}{(x^2+4)^2} \end{aligned}$$

D) The inflection points of this function are at the roots of the numerator of $f''(x)$ – $x = 0, \pm\sqrt{12}$. The second derivative changes sign at each of these.

III. Let x, y be the overall dimensions of the poster. The printed region is $x - 4$ inches by $y - 3$ inches because of the margins. Then we have $xy = 400$, and we want to maximize the area of the printed region, which is $A = (x-4)(y-3)$. Solving for y from the equation $xy = 400$, $y = 400/x$, so

$$\begin{aligned} A(x) &= (x-4)\left(\frac{400}{x} - 3\right) \\ &= 400 - \frac{1600}{x} - 3x + 12 \\ &= 412 - \frac{1600}{x} - 3x \\ A'(x) &= \frac{1600}{x^2} - 3 \end{aligned}$$

The critical points of the area function are at the solutions of $A'(x) = 0$, so

$$\begin{aligned} \frac{1600}{x^2} - 3 &= 0 \\ \frac{1600}{x^2} &= 3 \\ x^2 &= \frac{1600}{3} \\ x &= \sqrt{\frac{1600}{3}} \\ &\doteq 23.1 \end{aligned}$$

Then $y = \frac{400}{23.1} \doteq 17.3$. This is a maximum for the area because $A''(x) = -\frac{3200}{x^3}$, so $A''\left(\sqrt{\frac{1600}{3}}\right) < 0$ (Second Derivative Test says that if f'' is negative at the critical point, then f has a local maximum there).

IV.

A) *True*: Say $p_1 < p_2$. If we assume that f' is continuous for all x, then f' must be zero somewhere between p_1 and p_2. If not, f' would have to have the same sign everywhere between p_1 and p_2. That sign would have to be negative because p_1 is a local max. But $f'(x)$ cannot be negative to the left of p_2 if p_2 also a local max.

B) *True*: By implicit differentiation, $2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$, so

$$\frac{dy}{dx} = \frac{-y}{2y+x}$$

In this case we can solve explicitly for y as a function of x with the quadratic formula:

$$y = \frac{-x \pm \sqrt{x^2+4}}{2}$$

Each of these functions is defined for all x, and $2y + x = \sqrt{x^2+4} \neq 0$ for all x. Hence $\frac{dy}{dx}$ exists.