MATH 392 – Geometry Through History – Solutions for Quiz 2 – April 15, 2016

Let S be the surface parametrized by

$$\vec{x}(u,v) = (u, \cosh(u)\cos(v), \cosh(u)\sin(v)), \quad (u,v) \in (-\infty,\infty)\times(-\infty,\infty)$$

where the function $\cosh(u) = \frac{e^u+e^{-u}}{2}$ is the hypberbolic cosine.

(A) (5) Compute the coefficients E, F, G of the first fundamental form.

Solution: We have

$$\begin{aligned}
\vec{x}_u &= (1, \sinh(u)\cos(v), \sinh(u)\sin(v)) \\
\vec{x}_v &= (0, -\cosh(u)\sin(v), \cosh(u)\cos(v)) \\
E &= \langle \vec{x}_u, \vec{x}_u \rangle = 1 + \sinh^2(u)(\cos^2(v) + \sin^2(v)) = \cosh^2(u) \\
F &= \langle \vec{x}_u, \vec{x}_v \rangle = 0 \\
G &= \langle \vec{x}_v, \vec{x}_v \rangle = \cosh^2(u)(\sin^2(v) + \cos^2(v)) = \cosh^2(u).
\end{aligned}$$

(B) (10) Compute the coefficients e, f, g of the second fundamental form.

Solution: From part (A), the unit normal is computed like this:

$$\vec{x}_u \times \vec{x}_v = (\sinh(u)\cosh(u), -\cosh(u)\cos(v), -\cosh(u)\sin(v))$$

and

$$\|\vec{x}_u \times \vec{x}_v\| = \sqrt{\cosh^2(u)(1+\sinh^2(u))} = \cosh^2(u).$$

So

$$N = \left(\frac{\sinh(u)}{\cosh(u)}, -\frac{\cos(v)}{\cosh(u)}, -\frac{\sin(v)}{\cosh(u)}\right).$$

We have

$$\begin{aligned}
\vec{x}_{uu} &= (0, \cosh(u)\cos(v), \cosh(u)\sin(v)) \\
\vec{x}_{uv} &= (0, -\sinh(u)\sin(v), \sinh(u)\cos(v)) \\
\vec{x}_{vv} &= (0, -\cosh(u)\cos(v), -\cosh(u)\sin(v)) \\
e &= \langle \vec{x}_{uu}, N \rangle = -1 \\
f &= \langle \vec{x}_{uv}, N \rangle = 0 \\
g &= \langle \vec{x}_{vv}, N \rangle = 1.
\end{aligned}$$

(C) (5) Find the Gaussian curvature of S.

Solution: The Gaussian curvature is

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-1}{\cosh^4(u)}.$$