MATH 392 – Geometry Through History
Midterm Exam Solutions
March 22, 2016

I. Short Answer. Answer any four (4) of the following briefly (two or three sentences will suffice for each). Only the best 4 will be used to compute your total score.

(A) (5) Approximately when and where was Euclid active? How much do we know about his life?

Solution: Euclid lived around 300 BCE, in Alexandria in Egypt for a part of his life, at least. Almost nothing else about his life is known besides a tradition that he was connected in some way with Plato's school of philosophy and a few anecdotes about his teaching. (Most of this is deduced from a statement of the commentator Proclus from about 750 years later.)

(B) (5) State the 5 Postulates for geometry Euclid included at the start of Book I of the *Elements*. What was the function of these statements in Euclid's development of geometry?

Solution: Postulate I: A straight line segment can be drawn between any two points. Postulate II: A straight line segment can be extended indefinitely in both directions. Postulate III: A circle can be constructed given any point as the center and given any line segment as radius. Postlate IV: All right angles are equal. Postulate V: If a line falling on two other lines makes the angles on one side less than two right angles, then the two lines if extended will meet on the side with the angles summing to less than two right angles. These were unproved starting assumptions, from which Euclid wanted to deduce all known facts about geometry. (One could also say that it was pretty clear that Euclid thought of these as intuitively obvious facts about the geometry of physical space.)

(C) (5) What was Proclus's opinion about Euclid's Postulate V? How was this train of thought continued in later work?

Solution: Proclus thought that Postulate V was too complicated and unobvious to be a postulate. He wanted to prove it as a theorem from the other Postulates and results derived from them. This line of attack was taken up by many other mathematicians up until the late 18th century CE.

(D) (5) Approximately when and where was Girolamo Saccheri, S.J. active? What were his main contributions?

Solution: Saccheri lived from 1667 to 1733 in Italy. He attempted to prove Postulate V arguing by contradiction using the properties of figures that we now call Saccheri quadrilaterals. He proved many theorems about hyperbolic geometry in the process in his book *Euclid Freed of Every Flaw*, but he eventually declared that he had found a contradiction to the nature of a straight line and "punted."

(E) (5) Why is Gauss now always included with Bolyai and Lobachevsky in discussions of the developers of hyperbolic geometry when he did not publish any work on that subject?

Solution: He was one of the first mathematicians to realize that no contradiction would result from a geometry where Euclid's Postulate V did not hold. He developed many of the theorems of hyperbolic geometry independently and before Bolyai and Lobachevski. But he decided not to publish his work because he did not think it would be understood and he wanted to avoid controversy and preserve his reputation(!)

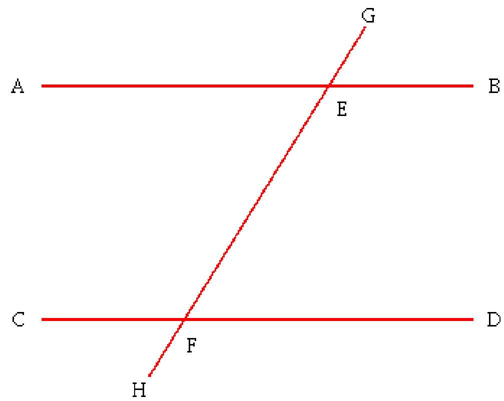

Figure 1: Figure for Euclid I.29

(F) (5) What does Pasch's Axiom say and why do we include that statement in a complete set of axioms for Euclidean geometry?

Solution: Pasch's Axiom says that a line that enters a triangle through a point on one or two of the sides must also exit the triangle at a point one of the sides not already crossed. It is necessary to include this as an axiom to provide complete proofs of several propositions.

II. (20) State and prove Proposition 29 in Book I of the *Elements*. What is especially notable about this proposition? (Note: If you need to, you may "buy" the statement to be proved from me for 5 of the possible points on this problem.)

Solution: Refer to the figure above. The statement is: If lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel and line $\overleftrightarrow{GH}$ is a transversal, then (a) alternate interior angles are equal, (b) corresponding angles are equal, and (c) the interior angles on one side of the transversal add to two right angles.

Proof: We prove (c) first and deduce the other two statements from that one. Suppose $\angle EFD + \angle FEB <$ two right angles. Then Postulate V implies that the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ extended must meed on that side of the transversal $\overleftrightarrow{GH}$. But this cannot be true since we assumed $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel. Similarly, if $\angle EFD + \angle FEB >$ two right angles, then $\angle CFE + \angle AEF$ must be less than two right angles and we do the same argument on that side. It follows that $\angle EFD + \angle FEB =$ two right angles. Now for (b), corresponding angles are ones like the pair $\angle GEB$ and $\angle EFD$. By a previous proposition (I.13 to be exact), $\angle GEB + \angle FEB =$ two right angles since G, E, F are collinear. But then from (c) we also have $\angle EFD + \angle FEB =$ two right angles. Hence $\angle GEB = \angle EFD$ by Common Notion 3 ("if equals are subtracted from equals, the remainders are equal"). Finally for (a), opposite interior angles are angles like the pair $\angle AEF$ and $\angle EFD$. We have $\angle GEB = \angle AEF$ by I.15 (equal vertical angles). Then from (b) and Common Notion 1, we have the desired statement $\angle AEF = \angle EFD$.

This was the first use made of Postulate V in Book I of the *Elements*.

III. Definitions and related facts.

(A) (10) What is the *angle of parallelism* for a segment AP in hyperbolic geometry? What does the Bolyai-Lobachevsky theorem tell us about the angle of parallelism?

Solution: Let $\overline{AP} \perp \overline{PQ}$. The ray line $\overrightarrow{AB}$ from A is said to be parallel to $\overrightarrow{PQ}$ in the direction of Q, B if for any point S in the quadrilateral $PQBA$, the ray $\overrightarrow{AS}$ intersects $\overrightarrow{PQ}$, but $\overrightarrow{AB}$ does not meet $\overrightarrow{PQ}$. The angle of parallelism $\Pi(AP)$ is the angle $\angle PAQ$. The Bolyai-Lobachevsky theorem says that this angle, which depends only on the length $x = AP$ can be computed by

$$\tan(\Pi(x)/2) = e^{-x/k}$$

for some constant k.

(B) (10) What is a *horocycle* in hyperbolic geometry? What is the closest Euclidean analog to this curve?

Solution: Consider a pencil of parallel lines. If P is a given point, then the horocycle through P is the locus of points Q such that Q corresponds to P with respect to the pencil. (In other words, the line of the pencil passing through the midpoint of the line segment $\overline{PQ}$ is perpendicular to $\overline{PQ}$. The closest Euclidean analog is probably the circle, but the center of the circle would be a point "at infinity."

IV. Refer to the figure at the top of the next page. A *median* of a triangle is the line segment joining a vertex to the midpoint of the opposite side. In the figure, $\overline{AA'}$, $\overline{BB'}$, and the dotted line $\overline{CC'}$ are medians. In this problem, from the parts below you will find a proof of the theorem that

The three medians of a triangle meet in a single point (called the centroid of the triangle).

Let O be the point where two medians meet as in the figure[1] The idea of this proof is to show that

$$(*)\ O \text{ is a point such that } AO = 2 \cdot OA' \text{ and } BO = 2 \cdot OB'.$$

In other words, (any) two medians *cut each other* at a point O in two line segments in the ratio $1 : 2$, with the shorter distance being the distance from O to the midpoint of the opposite side.

(A) (5) Explain why establishing (*) is enough to show that the three medians all intersect at O. (Don't think too hard about this – it's very easy!)

Solution: If (*) is true then the third median $\overline{CC'}$ must also pass through the point O because there is only one point on each median that cuts the median into two line segments in the ratio $1 : 2$.

(B) (5) Show that area$(\Delta ABA')$ = area$(\Delta ACA')$, area$(\Delta OBA')$ = area$(\Delta OCA')$, and similarly area$(\Delta BAB')$ = area$(\Delta BCB')$ and area$(\Delta OAB')$ = area$(\Delta OCB')$.

Solution: These all follow from the statement of I.38 (triangles on equal bases and in the same parallels have equal area). A more modern way to say this is that each pair of triangles has equal bases and equal altitudes, so the area formula $\frac{1}{2}bh$ shows the areas are the same.

[1] *Comment*: Such a point O must exist by Pasch's Axiom, applied to the triangle $\Delta AA'B$ and the line $\overleftrightarrow{BB'}$.

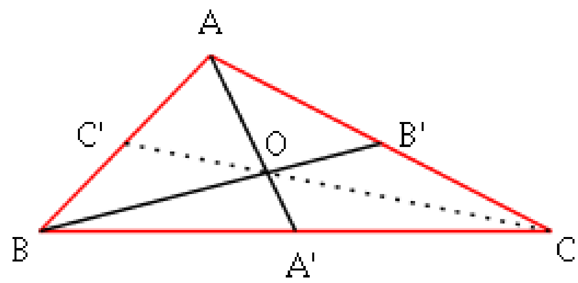

Figure 2: ΔABC with medians AA' and BB' in solid black

(C) (5) Deduce that

$$\text{area}(\Delta OBA') = \text{area}(\Delta OCA') = \text{area}(\Delta OCB') = \text{area}(\Delta OAB').$$

Solution: We already know that the first two of these are equal and the last two of them are equal by (B). So we need to show that one of the first pair is equal to one of the second pair. Note that since $\text{area}(\Delta ABA') = \text{area}(\Delta ACA')$, each of these triangles has half the area of ΔABC. Since $\text{area}(\Delta BAB') = \text{area}(\Delta BCB')$, the same is true for those two triangles. It follows that

$$\text{area}(\Delta BAB') = \text{area}(\Delta BOA) + \text{area}(\Delta OAB') = \text{area}(\Delta ABA') = \text{area}(\Delta BOA) + \text{area}(\Delta OBA').$$

Hence $\text{area}(\Delta OAB') = \text{area}(\Delta OBA')$. Then all four of those triangles are equal in area.

(D) (5) Finally, show that $AO = 2 \cdot OA'$ and $BO = 2 \cdot OB'$, which is the desired statement (*) above.

Solution: Look at parts (B) and (C). The triangle $\Delta ACA'$ is split into three equal smaller triangles. The triangle $\Delta ABA'$ has the same area and is split into $\Delta OBA'$ (which is equal in area to each of the three smaller triangles) and ΔOAB. This says

$$\text{area}(\Delta OAB) = 2 \cdot \text{area}(OBA').$$

But now those two triangles have bases along the same line and a common vertex at B, so the bases satisfy $AO = 2 \cdot OA'$. The statement $BO = 2 \cdot OB'$ follows similarly by considering the triangles $\Delta BAB'$ and $\Delta BCB'$.

V. Consider the figure at the top of the next page.

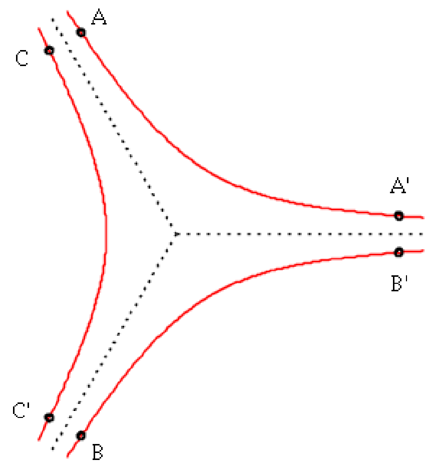

Figure 3: An "asymptotic triangle" in the hyperbolic plane

(A) (15) Show that in the hyperbolic plane it is possible to construct three pairwise parallel hyperbolic lines as in the figure – that is show how to construct three lines ℓ_1, ℓ_2, and ℓ_3 and points A, A' on ℓ_1, B, B' on ℓ_2 and C, C' on ℓ_3 such that $\overrightarrow{AA'}$ is parallel to $\overrightarrow{BB'}$ in the direction of A' and B', and (at the same time) $\overrightarrow{B'B}$ is parallel to $\overrightarrow{CC'}$ in the direction of B and C', and (at the same time) $\overrightarrow{A'A}$ is parallel to $\overrightarrow{C'C}$ in the direction of A and C.

Solution: Recall from Problem Set 3 that given any angle $\angle AOB$, less than two right angles, there is a line $\overleftrightarrow{PQ}$ that such that $\overrightarrow{QP}$ is parallel to $\overrightarrow{OA}$ in the direction of P and A and also $\overrightarrow{PQ}$ is parallel to $\overrightarrow{OB}$ in the direction of Q and B. The construction was as follows. Let $\overrightarrow{OD}$ be the angle bisector of $\angle AOB$ and let D be a point along that bisector such that the angle of parallelism $\Pi(OD) = \angle AOD = \angle BOD$. Then the line perpendicular to $\overrightarrow{OD}$ through D is the line. To get the lines as in the figure, take three rays meeting at a point O at pairwise angles of $120°$ (for instance – any three angles summing to $360°$ with none $\geq 180°$ would also work). Apply the result from the problem set to each pair of rays to get three lines. The only remaining thing to show is that the lines are pairwise parallel. But that follows since for instance once we know $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ are both parallel to the ray shown as dotted black in the figure, then they must also be parallel to each other.

(B) (5) Generalize your argument to show that there are similar "asymptotic n-gons" of parallel lines for any $n \geq 3$.

Solution: Apply the same result from the Problem Set 3 to n rays meeting at angles of $\frac{360°}{n}$.