MATH 392 – Geometry Through History
Midterm Exam
March 22, 2016

Instructions: Do all work in the blue exam booklet. There are 100 possible points.

I. Short Answer. Answer any four (4) of the following briefly (two or three sentences will suffice for each). Only the best 4 will be used to compute your total score.

(A) (5) Approximately when and where was Euclid active? How much do we know about his life?

(B) (5) State the 5 Postulates for geometry Euclid included at the start of Book I of the *Elements*. What was the function of these statements in Euclid's development of geometry?

(C) (5) What was Proclus's opinion about Euclid's Postulate V? How was this train of thought continued in later work?

(D) (5) Approximately when and where was Girolamo Saccheri, S.J. active? What were his main contributions?

(E) (5) Why is Gauss now always included with Bolyai and Lobachevsky in discussions of the developers of hyperbolic geometry when he did not publish any work on that subject?

(F) (5) What does Pasch's Axiom say and why do we include that statement in a complete set of axioms for Euclidean geometry?

II. (20) State and prove Proposition 29 in Book I of the *Elements*. What is especially notable about this proposition? (Note: If you need to, you may "buy" the statement to be proved from me for 5 of the possible points on this problem.)

III. Definitions and related facts.

(A) (10) What is the *angle of parallelism* for a segment AP in hyperbolic geometry? What does the Bolyai-Lobachevsky theorem tell us about the angle of parallelism?

(B) (10) What is a *horocycle* in hyperbolic geometry? What is the closest Euclidean analog to this curve?

IV. Refer to the figure at the top of the next page. A *median* of a triangle is the line segment joining a vertex to the midpoint of the opposite side. In the figure, $\overline{AA'}$, $\overline{BB'}$, and the dotted line $\overline{CC'}$ are medians. In this problem, from the parts below you will find a proof of the theorem that

The three medians of a triangle meet in a single point (called the centroid of the triangle).

Let O be the point where two medians meet as in the figure. The idea of this proof is to show that

(*) O is a point such that $AO = 2 \cdot OA'$ and $BO = 2 \cdot OB'$.

In other words, (any) two medians *cut each other* at a point O in two line segments in the ratio $1 : 2$, with the shorter distance being the distance from O to the midpoint of the opposite side.

(A) (5) Explain why establishing (*) is enough to show that the three medians all intersect at O. (Don't think too hard about this – it's very easy!)

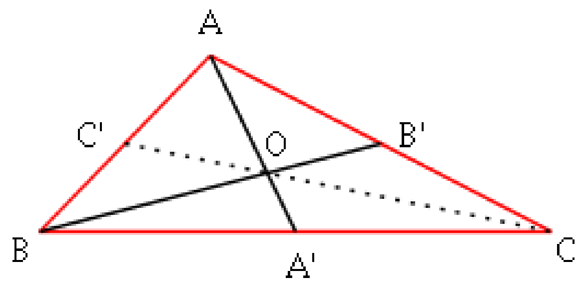

Figure 1: ΔABC with medians AA' and BB' in solid black

(B) (5) Show that area($\Delta ABA'$) = area($\Delta ACA'$), area($\Delta OBA'$) = area($\Delta OCA'$), and similarly area($\Delta BAB'$) = area($\Delta BCB'$) and area($\Delta OAB'$) = area($\Delta OCB'$).

(C) (5) Deduce that

$$\text{area}(\Delta OBA') = \text{area}(\Delta OCA') = \text{area}(\Delta OCB') = \text{area}(\Delta OAB').$$

(D) (5) Finally, show that $AO = 2 \cdot OA'$ and $BO = 2 \cdot OB'$, which is the desired statement (*) above.

V. Consider the figure on page 3.

(A) (15) Show that in the hyperbolic plane it is possible to construct three pairwise parallel hyperbolic lines as in the figure – that is show how to construct three lines $\ell_1, \ell_2,$ and ℓ_3 and points A, A' on ℓ_1, B, B' on ℓ_2 and C, C' on ℓ_3 such that $\overrightarrow{AA'}$ is parallel to $\overrightarrow{BB'}$ in the direction of A' and B', and (at the same time) $\overrightarrow{B'B}$ is parallel to $\overrightarrow{CC'}$ in the direction of B and C', and (at the same time) $\overrightarrow{A'A}$ is parallel to $\overrightarrow{C'C}$ in the direction of A and C.

(B) (5) Generalize your argument to show that there are similar "asymptotic n-gons" of parallel lines for any $n \geq 3$.

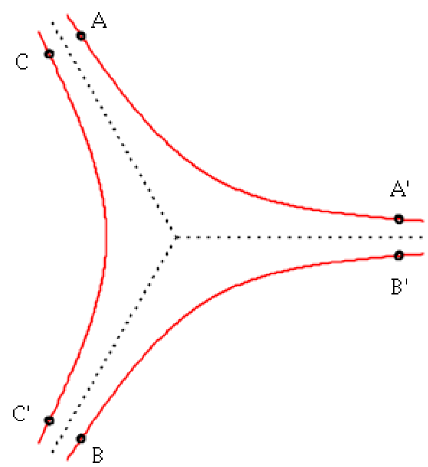

Figure 2: An "asymptotic triangle" in the hyperbolic plane