Mathematics 357 – Combinatorics
Discussion 1 – The Game of SET
January 25 and 27, 2017

Background and Goals

The card game called SET^{TM} is played with a deck of 81 cards, one with each possible combination of four attributes:

- A *number*: 1, 2, or 3
- A *shape*: oval, diamond, or "squiggle"
- A *color*: red, green, or purple
- A *shading*: open, shaded, solid

If we make some (any) associations of the numbers, shapes, colors, and shadings with the set $\{0, 1, 2\}$, then we can think of the deck of cards as represented by the set $\{0, 1, 2\}^4$ of ordered 4-tuples (a_1, a_2, a_3, a_4), where $a_1 \in \{0, 1, 2\}$ gives the number, a_2 the shape, etc.

A "set" in the terminology of the game is a collection of exactly *three cards* that has the property that with respect to each of the four attributes, the three cards are either *all the same*, or else *all different*. We will always use the quotation marks to distinguish "sets" in this sense from sets in the more general mathematical sense.

In today's discussion, we want to take this setup from the game of SET and look at a sampling of different types of combinatorial questions that one might ask concerning the SET deck and the special arrangements of cards called "sets".

A) Some "warmup" *enumeration* questions.

1) Let **2** denote the set of all cards showing number 2, let **Green** denote the set of all red-colored cards, and let **Squiggle** denote the set of all "squiggle"-shaped cards. Say in words what the following sets represent, and find the number of cards in each:

$$A = \mathbf{2} \cap \mathbf{Green}, \quad B = \mathbf{Squiggle} - \mathbf{Green}, \quad C = (\mathbf{2} \cap \mathbf{Squiggle}) \cup \mathbf{Green}$$

(B is the *set difference*).

B) One of the interesting things about combinatorics is the way an *algebraic* structure often underlays the arrangements in a combinatorial problem. (This is one reason why MATH 357 satisfies the Algebra breadth area for Mathematics majors!). We will pick and fix an identification of the Set deck of 81 cards with the set $\{0, 1, 2\}^4$ as discussed last time. What algebraic operations can you do with the elements of $\{0, 1, 2\}$? (Think about ideas you learned in Algebraic Structures.) What is always true about the *vector sum* of the three four-tuples corresponding to the three cards in a "set"? Prove your assertion.

C) Use your answer to B to answer the following additional enumeration questions, and justify your reasoning with a complete explanation. (Note: These can also be done by "brute force" if necessary, and that's a good check on your answers. But you should find that using question B makes these *much easier*!)

1) Given any two distinct cards in the deck, how many "sets" contain both of them?
2) How many "sets" are there in the deck that contain a single, given card? (Hint: the answer is the same for all cards, but you'll need to be careful in counting so that you don't end up counting the same "set" more than once.)
3) How many "sets" are there in the entire deck?

D) Other combinatorial questions deal with the *existence* of certain arrangements. Here's a question of this type. In the play of the game, is it possible that the 12 cards dealt contain *no "sets" at all*? Find such a set of 12 cards, or say why none exists.

E) On the other hand, it's possible for 12 cards to contain *lots* of "sets". In fact, the Puzzle page of the online edition of the *New York Times* contains daily SET puzzles where 12 given cards contain exactly 6 "sets." See `http://www.nytimes.com/crosswords/game/set/` and look at the two "Advanced" puzzles for examples. Give a general procedure to construct examples where 12 distinct cards contain (at least) 6 different "sets." (Hint: One approach is to pick some subset of the cards at random and then "fill in" the rest to make "sets" using what you saw in question B.) Does your method always produce *exactly* 6 "sets" or can you get more of them? Can you restrict the choices involved so that you get *only* 6 different "sets?"

Assignment

Group writeups due Wednesday, February 1.