College of the Holy Cross, Fall Semester, 2015
MATH 133, section 1, Solutions for Midterm 4
Thursday, December 10

1. Find $\frac{dy}{dx}$; do not simplify:

   A) (7.5) $y = \ln(\sin(x) + \cos(x))$

   By the derivative rule for $\ln(x)$ and the chain rule:

   $$\frac{dy}{dx} = \frac{1}{\sin(x) + \cos(x)} \cdot (\cos(x) - \sin(x))$$

   B) (7.5) $y = \tan^{-1}(e^{3x})$

   By the derivative rule for $\tan^{-1}(x)$ and the chain rule:

   $$\frac{dy}{dx} = \frac{1}{1 + (e^{3x})^2} \cdot e^{3x} \cdot 3.$$

   C) (10) $x^2y^4 - 4\sin^{-1}(y^2) + x = 0$ (use implicit differentiation)

   Differentiating with respect to x, thinking of y as a function of x:

   $$4x^2y^3\frac{dy}{dx} + 2xy^4 - \frac{4}{\sqrt{1-(y^2)^2}} \cdot 2y \cdot \frac{dy}{dx} + 1 = 0.$$

   Then solving for $\frac{dy}{dx}$:

   $$\frac{dy}{dx} = \frac{-2xy^4 - 1}{4x^2y^3 - \frac{8y}{\sqrt{1-y^4}}}$$

   (this could be simplified of course, but the directions said not to do that, so ...)

2. (20) A stationary observer watches a weather balloon being launched from a point 1000 feet away from her position. The balloon rises at a rate of 20 feet per second. How fast is the distance between the balloon and the observer changing when the balloon is 400 feet above the ground?

   Solution: Let y be the distance of the balloon above the ground and z be the distance from the balloon to the observer. Since the position of the observer, the point the balloon was launched, and the position of the balloon form a right triangle at all times, the Pythagorean theorem says

   $$y^2 + 1000^2 = z^2.$$

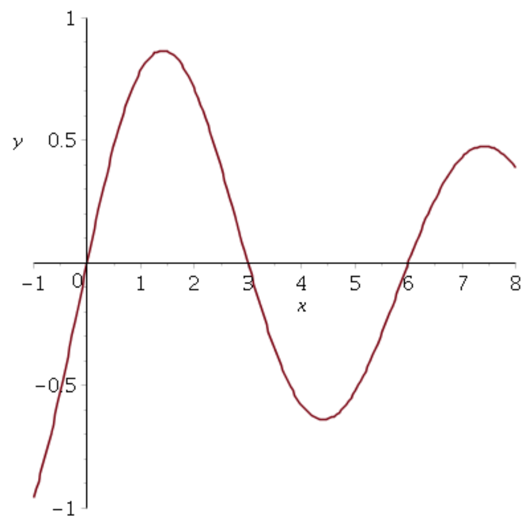

Figure 1: $y = f'(x)$ for Problem 3

Differentiating with respect to time, and cancelling factors of 2

$$y\frac{dy}{dt} = z\frac{dz}{dt}.$$

When $y = 400$, $z^2 = 400^2 + 1000^2$, so $z \doteq 1077$ feet and then $\frac{dy}{dt} = 20$ ft/sec is given. So

$$\frac{dz}{dt} \doteq \frac{400 \cdot 20}{1077} \doteq 7.43$$

feet per second.

3. All parts of this question refer to the plot in Figure 1, *which is $y = f'(x)$ for some function $f(x)$* . Assume the whole domain of the functions $f(x)$ and $f'(x)$ is the interval $[-1, 8]$ shown.

   (A) (10) Give approximate values for the critical points of $f(x)$ in the interval shown: Answer: $x = 0, 3, 6$

   (B) (5) Briefly, in your own words, state how the First Derivative Test distinguishes between local maxima, local minima, and critical points that are neither:

   Anything like this is OK: If f' changes sign from positive to negative at $x = c$, then $f(c)$ is a local maximum value; if f' changes sign from negative to positive at $x = c$, then $f(c)$ is a local minimum value; if f' does not change sign, then $f(c)$ is neither.

   (C) (5) Identify each of the points you found in part (A) as a local maximum, local minimum, or neither:

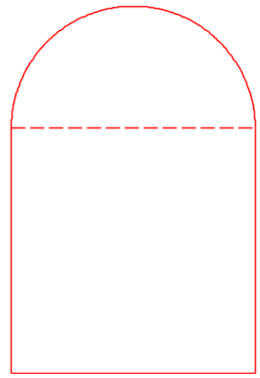

Figure 2: The window for Problem 4.

Answer: f has a local minimum at $x = 0$, a local maximum at $x = 3$ and another local minimum at $x = 6$.

(D) (5) Find approximate values for all the inflection points of $f(x)$.
Answer: $x \doteq 1.5, 4.5, 7.5$ (the points where $f''(x) = 0$ and f'' changes sign).

(E) (5) Over which intervals is $y = f(x)$ concave up? concave down?
Concave up: $(-1, 1.5) \cup (4.5, 7.5)$ (where f' is increasing) Concave down: $(1.5, 4.5) \cup (7.5, 8)$ (where f' is decreasing).

4. A church window has the shape of a rectangle surmounted by a semicircle (see Figure 2). The total perimeter of the window is 800 cm. What should the dimensions be to make the area of the window be as large as possible (so that it will admit the most light possible)? Useful information: The area of a circle of radius r is $A = \pi r^2$ and the circumference is $C = 2\pi r$. The area of a rectangle is the product of its length and width. The parts of this problem will lead you to the answer.

(A) (5) Call the horizontal side of the rectangle x and the vertical side y. Express the total area of the window and the total perimeter in terms of x and y.

The total area is the area of the rectangle plus the area of the semicircle. *To solve this one successfully, you must note that the radius of the semicircle is half the horizontal side of the rectangle:* $\frac{x}{2}$. So the total area of the window is

$$A = xy + \frac{1}{2} \cdot \pi \left(\frac{x}{2}\right)^2 = xy + \frac{\pi x^2}{8}$$

and the total perimeter is the given 800 cm, made up of one x (not two because the semicircle is joined to the rectangle along the dotted line so that side is not counted in the perimeter), two y's and half the circumference of the circle:

$$P = 800 = x + 2y + \frac{1}{2} \cdot 2\pi \cdot \frac{x}{2} = x + 2y + \frac{\pi x}{2}$$

(B) (5) Solve for y in terms of x using the perimeter equation and substitute into the area equation.

We have

$$y = \frac{1}{2}\left(800 - x - \frac{\pi x}{2}\right) = 400 - \frac{x}{2} - \frac{\pi x}{4}.$$

Substituting into the area function we get

$$A(x) = x\left(400 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{\pi x^2}{8} = 400x - \frac{x^2}{2} - \frac{\pi x^2}{8}.$$

(Note that I combined the two x^2 terms containing the π factors to make computing the derivative and finding critical points easier.)

(C) (5) Determine a critical point of the area function.

We differentiate and set the derivative equal to zero:

$$A'(x) = 400 - x - \frac{\pi x}{4} = 0.$$

Solving for x:

$$x = \frac{400}{1 + \pi/4} \doteq 224 \text{ cm}.$$

This is the only critical point of the area function.

(D) (5) How do you know your critical point is a maximum of the area?

The easiest way is to use the Second Derivative Test:

$$A''(x) = -1 - \frac{\pi}{4} < 0$$

is negative everywhere (including at the critical point). Therefore, the critical point is a local and absolute maximum for A. You could also use the First Derivative Test here.

(E) (5) What are the dimensions x and y of the window of maximum area?

We have $x = 224$ from (C) and (D) above. To get the value of y, we use the equation for y found in part (B) above:

$$y = 400 - \frac{x}{2} - \frac{\pi x}{4} \doteq 400 - 224/2 - \frac{\pi \cdot 224}{4} \doteq 112 \text{ cm}.$$