**College of the Holy Cross, Fall Semester, 2015**
**MATH 133, section 1, Midterm 1 Solutions**
**Thursday, September 24**

*Note:* a technical issue made it impossible to reproduce all of the graphs together in this document. The exam papers are also posted on the course homepage in case you want to look at the graphs that were given.

1. (a) (7.5) Express the set of all $x$ satisfying $|4x - 12| \leq 8$ as an interval or union of intervals.

   *Solution:* Algebraically, this inequality says $-8 \leq 4x - 12 \leq 8$, so $4 \leq 4x \leq 20$, so $1 \leq x \leq 5$. As an interval this is $[1, 5]$. Geometrically, we could get the same result by dividing by 4 to get $|x - 3| \leq 2$. The numbers at distance at most 2 along the number line from 3 are exactly $x \in [1, 5]$ as before.

   (b) (7.5) What is the domain of the function $f(x) = \frac{1}{x\sqrt{4-x}}$? Any correct form is OK.

   *Solution:* We must have $x \neq 0$ and $4 - x > 0$, or $x < 4$ The $x$'s satisfying both conditions form $(-\infty, 0) \cup (0, 4)$. Another correct way to say this: all real $x < 4$, except $x = 0$.

2. (15) The graph $y = f(x)$ and four graphs obtained by transforming it are shown. Match the given formulas with the corresponding graph. Note that there is an extra graph that does not match any of the formulas.

   (a) $y = f(\frac{1}{2}x)$ is graph (iv) – a horizontal stretch by a factor of 2
   (b) $y = \frac{1}{2}f(x)$: is graph (i) – a vertical compression by a factor of 1/2
   (c) $y = f(2x)$: is graph (iii).

   (The remaining graph (ii) is actually $y = \frac{1}{2} - f(x)$.)

3. (a) (10) Complete the square: $q(x) = 3x^2 + 18x + 36$.

   *Solution:* By the usual process,

$$q(x) = 3(x^2 + 6x + 12) = 3((x + 3)^2 + 3) = 9 + 3(x + 3)^2.$$

   (b) (5) What is the minimum value of $q(x)$?

   *Solution:* The smallest value attained by $q(x)$ is 9 since $3(x + 3)^2 \geq 0$ for all $x$.

(c) (5) Using your answers from parts (a) and (b), determine the *range* of the function $r(x) = \sqrt{q(x)} = \sqrt{3x^2 + 18x + 36}$.

*Solution:* Since $q(x) \geq 9$ for all $x$, $\sqrt{q(x)} \geq 3$ for all $x$. Moreover, all values $\geq 3$ are obtained for some $x$, so the range is the interval $[3, \infty)$.

4. The temperature $H$ in a desert varies sinusoidally from a high of 80°F at 5 PM to a low of 40°F at 5 AM. (See graph below)

(a) (5) What is the period of this sinusoidal oscillation? *Answer:* 24 hours

(b) (5) What is the amplitude? *Answer:* $\frac{1}{2}(80 - 40) = 20$

(c) (10) If $t = 0$ is the first 5PM time shown, give a possible formula for $H$ as a function of $t$.

*Solution:* Since the period starts at one of the maximum values this looks like a cosine graph, with horizontal and vertical scaling, plus a vertical shift. The vertical shift can be found from the average of the minimum and maximum values: $\frac{40+80}{2} = 60$. One form is

$$H(t) = 20\cos\left(\frac{2\pi t}{24}\right) + 60.$$

Another possibility would be

$$H(t) = 20\sin\left(\frac{2\pi(t+6)}{24}\right) + 60.$$

5. (a) (7.5) Solve for $x$: $4^{x+1} = 8^{x-3}$.

*Solutions:* The basic method: Take ln of both sides to get

$$(x+1)\ln(4) = (x-3)\ln(8).$$

Then isolate the $x$ with basic algebra:

$$x(\ln(4) - \ln(8)) = -3\ln(8) - \ln(4).$$

So

$$x = \frac{-3\ln(8) - \ln(4)}{\ln(4) - \ln(8)}.$$

This is one correct form, but it also can be simplified. Since $4 = 2^2$ and $8 = 2^3$, this simplifies to

$$x = \frac{-11\ln(2)}{-\ln(2)} = 11.$$

The clever method: Using laws of exponents, rewrite $4 = 2^2$ and $8 = 2^3$, so the equation becomes

$$2^{2(x+1)} = 2^{3(x-3)}.$$

This implies $2(x+1) = 3(x-3)$ or $2x + 2 = 3x - 9$, so $x = 11$. (!)

(b) (7.5) A sample of a radioactive element is decaying over time. The mass present at time $t$ is given by according to $M(t) = 139e^{-0.003t}$ grams, where $t$ is in months. When will the mass present reach 50 grams?

*Solution:* We want to solve the equation

$$50 = 139e^{-0.003t}$$

for $t$. Divide by 139 first to get

$$e^{-0.003t} = \frac{50}{139}.$$

Then taking natural logs of both sides gives

$$-0.003t = \ln\left(\frac{50}{139}\right) \Rightarrow t = \frac{\ln\left(\frac{50}{139}\right)}{-0.003} \doteq 340.8 \text{ months.}$$

6. You are traveling by donkey along a straight line road starting from $x = 0$ (miles) at time $t = 0$ (hours). For the first hour, you move in the positive $x$-direction at 5 miles per hour. At $t = 2$, you realize you have dropped an important item from your saddle bag. So you turn around and retrace your steps to 5 miles per hour. You find the item at $t = 3$. Then you turn back around and continue at 5 miles per hour for an additional 2 hours.

   (a) (7.5) Sketch the graph of your position $x$ as a function of time $t$ for $0 \leq t \leq 5$.

   (b) (7.5) Give your position $x$ as a piecewise-defined function of $t$.

*Solution:* Using the point-slope form for equation of lines, this is the function

$$x(t) = \begin{cases} 5t & \text{if } 0 \leq t \leq 2 \\ -5t + 20 & \text{if } 2 \leq t \leq 3 \\ 5t - 10 & \text{if } 3 \leq t \leq 5. \end{cases}$$

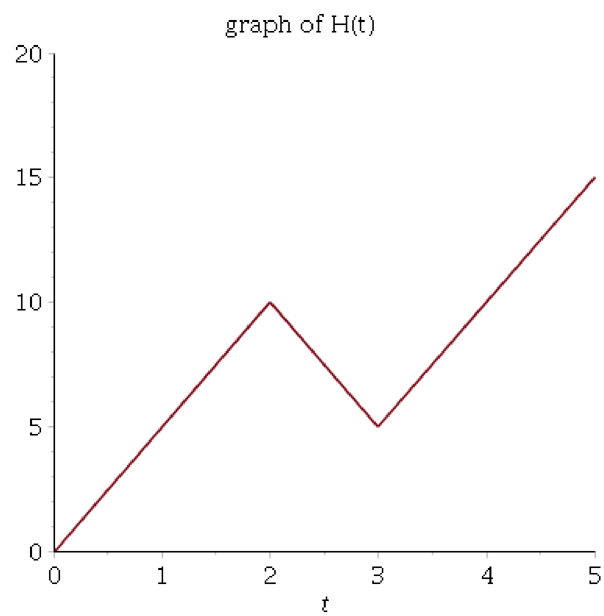


Figure 1: Figure for Question 6 (a)