MATH 133 – Calculus with Fundamentals 1
Discussion 1 – The Real Number System
September 3, 2015

Background

You are all familiar with decimal expansions of numbers such as

$$\frac{15}{8} = 1.875$$

Recall that in some cases, the decimal expansion must be continued indefinitely to get the exact value. For instance, we write

$$\frac{1}{3} = .33333\cdots = .\overline{3},$$

where the overline indicates that the digit 3 *repeats forever in the exact decimal expansion of* $\frac{1}{3}$. Another way of saying this is that if we stopped after any finite number of 3's the number we would have would be an *approximation* to $\frac{1}{3}$ but an *underestimate*:

$$.3333 = \frac{3333}{10000} < \frac{1}{3}.$$

The real number system is the set of *all numbers* that can be represented by these decimal expansions (possibly infinite, possibly nonrepeating).

Questions

A) How could you tell that $\frac{3333}{10000} < \frac{1}{3}$ just working with the fractions (i.e. without bringing the decimal expansions into it)?

B) What does the decimal expansion 1.875 for the fraction $\frac{15}{8}$ actually *mean*? (Hint: 1.875 is a way of writing this number as a *sum of several fractions* with "special" denominators. What is that sum?)

C) It is a fact that *every eventually repeating decimal expansion represents a number that can also be written as a single fraction with integer numerator and denominator* (a rational number). Let's see how this works by considering a representative example. For instance, consider the number

$$1.\overline{137} = 1.137137137\cdots$$

where the repeating part of the expansion starts right after the decimal point. We want to write this number in the form $\frac{a}{b}$ for some whole numbers (integers) a, b. If we start off by subtracting off the non-repeating whole number part, we get $1.\overline{137} - 1 = .\overline{137}$. Call this number c:

$$c = .137137137\cdots = .\overline{137}.$$

Note that
$$c - .137 = c - \frac{137}{1000} = .000137137\cdots = .000\overline{137}.$$
But this number can also obtained from c by multiplying by $\frac{1}{10^3} = \frac{1}{1000}$, so
$$c - \frac{137}{1000} = \frac{1}{1000}c.$$
(Make sure you see why this works by thinking about shifting the decimal point.) Use this equation to solve algebraically for c as a fraction, and then use that to express $1.\overline{137}$ as a fraction.

We can do something similar to this process for any eventually repeating decimal expansion.

D) Even if a decimal expansion *never* settles down into a repeating pattern, it still represents a real number. The numbers given by *nonrepeating decimal expansions* are called *irrational numbers*. You have probably seen several examples of important irrational numbers at some point – give a 6 decimal digit approximations of your "favorite" irrational number (using a calculator). Round to the nearest digit in the 6th decimal place. Is your approximations an overestimate or an underestimate of the exact value of your number?

E) Do you know a *geometrical* way to picture or think about the real number system? Describe how the connection between numbers and points of the geometric "model" of the real number system works. For instance, what choices do you need to make to set up the correspondence between numbers and points?