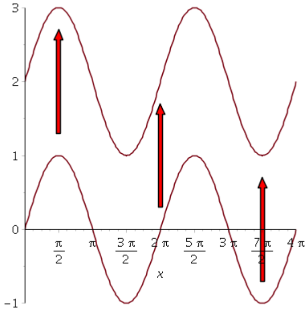
Shifting graphs; symmetries of graphs

MATH 133 – Calculus with Fundamentals, section 1, Prof. Little

June 10, 2015

Vertical shifts of graphs

Shift the whole graph $y = f(x) = \sin(x)$ up (i.e. along the direction of the positive y-axis) by 2 units in the xy-plane:



The new graph also satisfies the vertical line test (do you see why?) So the shifted graph is also the graph of some function $y = g(x)$. *What is the formula for $g(x)$ to get the shifted graph?*

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- This means the shifted graph comes from the function $y = g(x) = \sin(x) + 2$.
- The general pattern is the same and we can "work it both ways:" If we know the graph $y = f(x)$, then the graph $y = f(x) + c$ is a *vertical shift*, up by c units along the y-direction if $c > 0$, down by $|c|$ units if $c < 0$.

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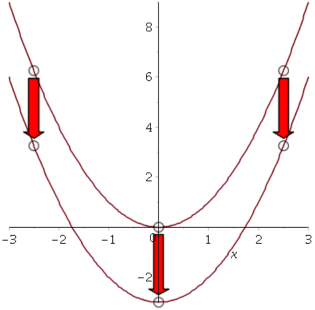
- This has the form $f(x) + c$ for $c = -3$.
- Since $-3 < 0$, we are shifting the whole parabola *down* by 3 units

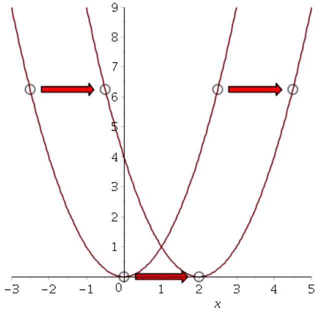
Shifted parabola



(Because of the way the parabola is curving, a bit of an *optical illusion* occurs here: it is not obvious that the vertical separation between the two graphs is always 3 units. But that can be seen from the points plotted as circles with the arrows.)

Horizontal shifting

Now, let's repeat the above sort of analysis, but shifting the graph $y = x^2$ by 2 units to the right.



For instance, from $(0, 0)$ on the original parabola, we get the corresponding point $(2, 0)$ on the shifted parabola.

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- For each point (a, a^2) on the original parabola $y = x^2$, we get a corresponding point $(a + 2, a^2)$ on the shifted parabola.
- Writing $x = a + 2$, we get $a = x - 2$ so $y = a^2 = (x - 2)^2$.

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- Since the points on the shifted parabola have the form $(x, (x - 2)^2)$, that is the graph of $y = (x - 2)^2$.
- By the same reasoning, shifting *left* by 1 unit would give the graph $y = (x + 1)^2$: If (a, a^2) is on $y = x^2$, then $(a - 1, a^2)$ is on the graph of $y = (x + 1)^2$ since if $x = a - 1$, then $a = x + 1$.

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- *But, be careful – this works differently from the vertical shifts*: The $+c$ is added to the x "*inside*" the function, *and* the signs of c work the *opposite way from what you might expect*: if $c > 0$ the shift is to the *left*, and if $c < 0$, the shift is to the *right*.

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- For example, $y = \sin(x - 1)$ is the graph $y = \sin(x)$ shifted to the *right* by one unit; while $y = \sin(x + \pi)$ is the sine graph shifted to the *left* by π units.

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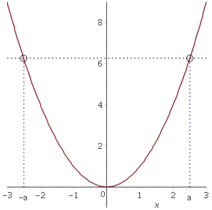
- The function $f(x) = x^2$ is an example since $f(-x) = (-x)^2 = x^2 = f(x)$ for all x.
- Saying f is even is the same as saying the graph $y = f(x)$ has *y-axis symmetry*.



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- *To visualize this*: If (a, b) is a point in the plane, then $(-a, -b)$ can be obtained by *rotating* the plane by 180 degrees (or π radians) about the origin.
- Rotating the whole plane this way, the graph of an odd function is *taken to itself* (but points in the first quadrant would get rotated around to the third quadrant, etc.).

An odd function and symmetry about the origin

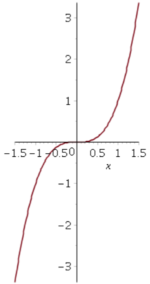

Figure: $y = x^3$

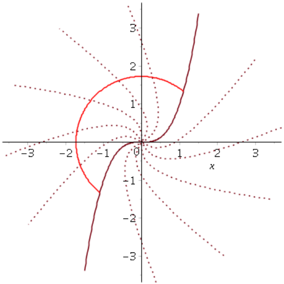

Figure: Rotations of $y = x^3$

On the right, the dotted curves are rotations of $y = x^3$ by whole number multiples of 30 degrees counterclockwise. The red arc gives the path followed by $(1.1, (1.1)^3)$ on $y = x^3$. After a rotation through 180 degrees, we're back to $y = x^3$(!)