Absolute Values and Intervals

MATH 133 – Calculus with Fundamentals, section 1, Prof. Little

June 10, 2015

Absolute Value

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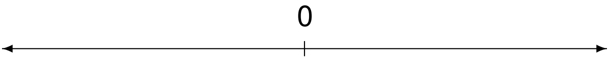
- but since $-8.3 < 0$,

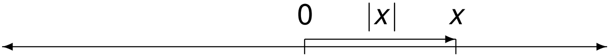
$$|-8.3| = -(-8.3) = +8.3.$$

Geometric meaning

We know that the collection of all real numbers can be seen as the set of points on the *number line*.



- If $x \geq 0$, then $|x|$ is the distance from 0 to x along the line:



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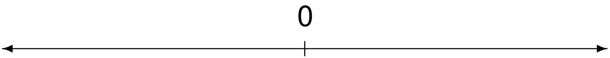

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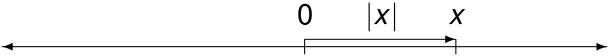
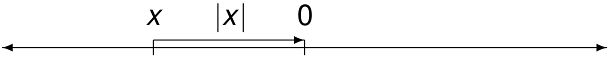
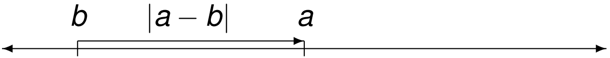


- If $x < 0$, $|x|$ is still the *positive* distance along the line:



Similarly, ...

If a and b are *any* two real numbers, then $|a-b|$ is the (positive) distance along the number line from a to b (or what is the same, from b to a, if the numbers satisfy $b < a$ as in the following picture):



For instance, $|(-3)-2| = |-5| = 5$ gives the distance from $b = -3$ to $a = 2$ along the number line.

Intervals

We will use this notation for intervals on the real number line:

- The **closed interval** $[a, b]$ is the set of all real x with $a \leq x$ and $x \leq b$. It is more common to write $a \leq x \leq b$ here, but you should always understand that this is the same as the two separate inequalities $a \leq x$ and $x \leq b$ given first.
- The **open interval** (a, b) is the set of all real x with $a < x$ and $x < b$, or $a < x < b$ (not including the endpoints $x = a, b$).
- The **half-closed interval** $[a, b)$ includes the endpoint a (and all x strictly between a and b), but not b.
- Similarly, the **half-closed interval** $(a, b]$ includes the endpoint b, but not a.

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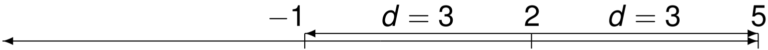

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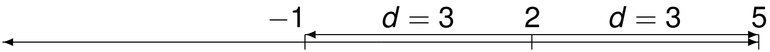

- That is, $|x - 2| < 3$ says the same thing as $-1 < x < 5$, so our set is the open interval $(-1, 5)$.

Another way to see this with algebra

For any real number a, the statement $|a| < r$ says the same thing as $-r < a < r$ (think of the distance from a to 0). So

$$\begin{aligned} |x-2| < 3 \quad & \text{is the same as} & -3 < x - 2 < 3, \\ & \text{which is the same as} & (-3) + 2 < x < 3 + 2, \\ & \text{which is the same as} & -1 < x < 5. \end{aligned}$$

So as before, the set of all such x is the interval $(-1, 5)$.

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- Conclusion: The interval $(4, 7)$ is the set of x with $|x - 5.5| < 1.5$.

The general pattern

If $a < b$, then the open interval (a, b):

- has midpoint $c = \frac{a+b}{2}$, and
- the distance from c to either endpoint is

$$\left| b - \frac{a+b}{2} \right| = \frac{b-a}{2} = \left| a - \frac{a+b}{2} \right|.$$

- So the interval is the set of all x with

$$\left| x - \frac{a+b}{2} \right| < \frac{b-a}{2}.$$

- No need to memorize this formula, but understand the process of how we got it!