Mathematics 126 – Calculus for Social Science 2
Solutions – Exam 3
December 3, 2001

I. This is a separable differential equation. To solve it, we can proceed as follows. First we factor .016 out of both terms on the right hand side:

$$\frac{dP}{dt} = .016(P - 210000/.016) = .016(P - 13125000)$$

Then divide by $P - 13125000$ and multiply by dt to separate. Integrating, then exponentiating both sides, we have:

$$\begin{aligned}
\int \frac{dP}{P - 13125000} &= \int .016\ dt \\
\ln|P - 13125000| &= .016t + c \\
P - 13125000 &= be^{.016t} \text{ (where } b = \pm e^c) \\
P &= 13125000 + be^{.016t}
\end{aligned}$$

When $t = 0$ (1847), $P = 8000000$, so $b = 8000000 - 13125000 = -5125000$. Then the population in 1855 ($t = 8$) is

$$P(8) = 13125000 - 5125000e^{(.016)(8)} \doteq 7300166.$$

(Because of the emigration, the total population of Ireland was decreasing over this time period.)

II.

A) The probability that x is in the range $0 \le x \le 1$ is

$$\begin{aligned}
P(0 \le x \le 1) &= \int_0^1 ce^{-cx}\ dx \\
&= -e^{-cx}|_0^1 \\
&= 1 - e^{-c}
\end{aligned}$$

Hence

$$\begin{aligned}
.1 &= 1 - e^{-c} \\
e^{-c} &= .9 \\
c &= -\ln(.9) \doteq .1053
\end{aligned}$$

B) For the expected value of x, we must take into account the *whole range of possible nonzero values of $f(x)$, which is $x \ge 0$.* Hence integrating by parts (or using #23 in the table of integrals):

$$\begin{aligned} E(x) &= \int_0^\infty xce^{-cx}\ dx \\ &= -xe^{-cx}|_0^\infty + \int_0^\infty e^{-cx}\ dx \\ &= 0 + \frac{-1}{c}e^{-cx}|_0^\infty \\ &= \frac{1}{c} \doteq 9.491 \text{ with } c = .1053 \text{ as above} \end{aligned}$$

C)

$$Var(x) = \int_0^\infty (x - 9.491)^2(.1053)e^{-.1053x}\ dx$$

III. The conversion from the car speed to the standard normal is $z = \frac{x-68}{5}$

A) The probability that a randomly selected car has speed $x \leq 65$ is:

$$\begin{aligned} P(x \leq 65) &= P\left(z \leq \frac{65-68}{5}\right) \\ &= P(z \leq -.6) \\ &\doteq .2743 \text{ (from the table)} \end{aligned}$$

B) The probability that a randomly selected car has speed $70 \leq x \leq 75$ is:

$$\begin{aligned} P(70 \leq x \leq 75) &= P(.4 \leq z \leq 1.4) \\ &= P(z \leq 1.4) - P(z \leq .4) \\ &\doteq .9192 - .6554 \text{ (from the table)} \\ &= .2638 \end{aligned}$$

IV.

A)

$$f_x = y - 4xy - 4y^2 \qquad f_y = x - 2x^2 - 8xy$$

$$f_{xx} = -4y \qquad f_{xy} = 1 - 4x - 8y \qquad f_{yy} = -8x$$

B) The critical points occur where:

$$\begin{aligned} f_x &= y(1 - 4x - 4y) = 0 \\ f_y &= x(1 - 2x - 8y) = 0 \end{aligned}$$

From these factored forms, we see either $y = 0$ or $4x + 4y = 1$, and either $x = 0$ or $2x + 8y = 1$. If $y = 0$, then either $x = 0$, or $2x = 1$, so we have two critical points with $y = 0$: $(0,0)$ and $(1/2,0)$. If $4x+4y=1$ and $x=0$, then $y=1/4$; if $4x+4y=1$ and $2x+8y=1$, then $(x,y)=(1/6,1/12)$ (subtract the first equation from 2 times the second).

For the second derivative test, we use the f_{xx}, f_{xy} and f_{yy} from part A, together with

$$H = f_{xx}f_{yy} - f_{xy}^2 = 32xy - (1-4x-8y)^2$$

We have

point	$H(x,y)$	f_{xx}	type
$(0,0)$	-1	0	saddle pt.
$(1/2,0)$	-1	0	saddle pt.
$(0,1/4)$	-1	-1	saddle pt.
$(1/6,1/12)$	$1/3$	$-1/3$	local max.

(For example,

$$H(1/6,1/12) = 32/72 - (1-2/3-2/3)^2 = 4/9 - 1/9 = 3/9 = 1/3)$$