Mathematics 174 – Applied Mathematics 2
Oops! – Revenge of the $2\pi$'s
April 11, 2000

In class today, I got an extra $2\pi$ thrown in with the frequency response function for the ideal "brick wall" low-pass filter (D'OH!!) Here is the way I wanted to say everything, before I got mixed up in class:

1) By the Big Theorem from class April 6, we know any filter is characterized by its impulse response: If we know $\gamma = \Gamma(\delta)$, then

$$g(x) = \Gamma(f)(x) = (f * \gamma)(x)$$

where $\gamma(x) = \Gamma(\delta)(x)$ is the *impulse response.*

2) By the convolution property of Fourier transforms, the Fourier transform $G$ of the output is related to the Fourier Transform $F$ of the input like this:

$$\begin{aligned} G(\omega) = \mathcal{F}[g](\omega) &= \mathcal{F}[f * \gamma](\omega) \\ &= 2\pi F(\omega)\mathcal{F}[\gamma](\omega) \\ &= H(\omega)F(\omega) \end{aligned}$$

where $H(\omega) = 2\pi\mathcal{F}[\gamma](\omega)$ is the *frequency response* – $2\pi$ times the Fourier transform of the impulse response.

3) For the ideal "brick wall" low-pass filter with cut-off $\omega_c$, we want to make the frequency response $H(\omega)$ so that the product $H(\omega)F(\omega)$ cuts out all the non-zero high-frequency values in $F$, but leaves $F$ unchanged in low frequencies. In other words, we really do want:

$$H(\omega) = \begin{cases} 1 & \text{if } |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

(1 on the pass-band, not $2\pi$!) This means that the Fourier transform of the impulse response is

$$\mathcal{F}[\gamma](\omega) = \frac{1}{2\pi}H(\omega) = \begin{cases} \frac{1}{2\pi} & \text{if } |\omega| \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Taking the inverse transform gives

$$\gamma(x) = \frac{1}{2\pi}\frac{2\sin(\omega_c x)}{x} = \frac{\sin(\omega_c x)}{\pi x}.$$

4) The magnitude of the frequency response $H(\omega)$ for the averaging filter *was correct* as given in the handout:

$$|H(\omega)| = \sqrt{\frac{2 - 2\cos(\omega)}{\omega^2}}$$