Mathematics 173 – Applied Mathematics 1
Lab/Problem Set 6 – November 18, 1999

*Goals*

In the lab portion of this assignment, we will compute and plot solutions to the 1D Wave Equation to visualize the corresponding motions of a string.

*Background*

Recall that last time we saw that the motions of a thin, highly stretched, perfectly elastic string can be modeled by the 1D Wave Equation: $u_{tt} = c^2 u_{xx}$, where $u$ is the vertical displacement from the equilibrium or rest position of the string.

*Lab Questions*

A) In class just now, we began to consider the solution of the boundary value problem:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = \begin{cases} 2ax/L & \text{if } 0 \leq x \leq L/2 \\ 2a(L-x)/L & \text{if } L/2 \leq x \leq L \end{cases} \\ u_t(x,0) = 0 \end{cases}$$

1) Using Maple, verify our formula for the solution

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8a \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

Take $a = .1, L = 10, c = 1$, and plot (partial sums of) this solution as functions of $x, t$. Describe the motion of the string. Look up the `animate` command in the `plots` package and see if you can use that to make an animation of the vibrating string.

2) Let $F$ be the odd periodic extension of the initial displacement function of the string $u(x,0)$ above (still with $a = .1, L = 10, c = 1$). Plot $F(x-ct)$ and $F(x+ct)$ as functions of $(x,t)$ using Maple. These are called *travelling waves*. Do you see why the name is appropriate?

3) What happens if you plot $(F(x-ct)+F(x+ct))/2$ in the interval $[0,10]$? Compare with your result from part 1. (Note: You will want to use Maple's `piecewise` mechanism to define the odd periodic extension. Of course, you cannot include an infinite number of different options in the definition. This means you will need to define a function that agrees with the odd periodic extension only on some interval, and then plot only $(x,t)$ such that $x + ct$ and $x - ct$ are in that

interval(!)) See problem D below for a general statement about what is happening here!

B) Repeat question A, parts 1,2 with the same initial string displacement function but now make $u_t(x,0) = x(L-x)/10$.

C) Note that the solutions in A,B represent strings that keep oscillating in an exact periodic fashion forever. In the real world, that never happens because of additional forces that we ignored in the derivation of the wave equation – e.g. air resistance, frictional forces at the ends, etc. In this problem, we will consider a PDE modeling a string moving in the presence of damping force(s) proportional to the string's velocity:

$$\rho_0 u_{tt} = T_0 u_{xx} - \beta u_t$$

where $\beta > 0$. *In addition, we assume that $\beta$ is relatively small:*

$$\beta^2 < 4\pi^2 \rho_0 T_0 / L^2.$$

Here $\rho_0$ is the mass density, and $T_0$ is the tension as before. We want to solve the boundary value problem

$$\begin{cases} \rho_0 u_{tt} = T_0 u_{xx} - \beta u_t \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = \begin{cases} 2ax/L & \text{if } 0 \le x \le L/2 \\ 2a(L-x)/L & \text{if } L/2 \le x \le L \end{cases} \\ u_t(x,0) = 0 \end{cases}$$

1) Show that if we apply the separation of variables method, looking for product solutions $u(x,t) = \phi(x)h(t)$, then we get the ordinary differential equations

$$\rho_0 h''(t) + \beta h'(t) + \lambda T_0 h(t) = 0$$

and

$$\phi''(x) = -\lambda \phi(x)$$

2) Show that $\phi(x) = \sin\left(\frac{n\pi x}{L}\right)$, and $\lambda = \left(\frac{n\pi}{L}\right)^2$ for some $n \ge 1$.
3) Show that if

$$w_n = \sqrt{\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2 - \frac{\beta^2}{4\rho_0^2}}$$

then

$$h(t) = e^{-\beta t/2\rho_0} \cos(w_n t)$$

and

$$h(t) = e^{-\beta t/2\rho_0} \sin(w_n t)$$

are both solutions of the equation for $h(t)$. Then we can construct series solutions:

$$u(x,t) = e^{-\beta t/2\rho_0} \sum_{n=1}^{\infty} (a_n \cos(w_n t) + b_n \sin(w_n t)) \sin\left(\frac{n\pi x}{L}\right)$$

Explain how to determine $a_n$ and $b_n$ to satisfy the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ for any given functions $f, g$.

D) Problem 4.4.7 from the text.

*Assignment*

Individual write-ups due Thursday, December 2.