Mathematics 173 – Applied Mathematics 1
Lab/Problem Set 4
October 19, 1999

Background and Review

Before the midterm, we worked out some first examples of solutions to boundary value problems for the 1D Heat Equation where the ends of the rod were held at zero temperature:

$$
\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \tag{1}
$$

Our method was the *separation of variables* technique:

1) We looked first for "product" solutions of the heat equation, that is: $u(x,t) = G(t)\phi(x)$ satisfying $u_t = k u_{xx}$. This led to the two ordinary differential equations:

$$\frac{dG}{dt} = -k\lambda G \quad \text{and} \quad \frac{d^2\phi}{dx^2} = -\lambda\phi,$$

from which we saw that

$$
\begin{aligned}
G(t) &= ce^{-k\lambda t} \\
\phi(x) &= \begin{cases} c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) & \text{if } \lambda > 0 \\ c_1 x + c_2 & \text{if } \lambda = 0 \\ c_1 \cosh(\sqrt{-\lambda}x) + c_2 \sinh(\sqrt{-\lambda}x) & \text{if } \lambda < 0 \end{cases}
\end{aligned} \tag{2}
$$

2) Solutions satisfying the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ are only possible in the case $\lambda > 0$, and we get the corresponding *eigenvalues* $\lambda = \left(\frac{n\pi}{L}\right)^2$ and *eigenfunctions* $\sin(\sqrt{\lambda}x) = \sin\left(\frac{n\pi x}{L}\right)$.
3) Then, to get a solution of (1) (including the initial condition $u(x,0) = f(x)$, we proceed as follows. If $f(x)$ is given by the *Fourier series expansion*

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right),$$

then our solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi x}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right). \tag{3}$$

4) The Fourier coefficients B_n are found by the following formula (consequence of the orthogonality relation for the functions $\sin\left(\frac{n\pi x}{L}\right)$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Today, we want to work out another example like the one we saw in class on 10/5 using Maple, then see how the method outlined above can *also* be used to solve other types of boundary value problems.

Lab and Problem Set Questions

I. In this question we will consider the BVP as in (1) with $k = 0.01$ (this means intuitively that heat energy diffuses relatively *slowly* through the material making up this rod), and $L = 10$:

$$(1)\qquad \begin{cases} u_t = (.01)u_{xx} \\ u(x,0) = 40e^{-10(x-3)^2} + 20e^{-20(x-8)^2} \\ u(0,t) = 0 \\ u(10,t) = 0 \end{cases}$$

You will follow what we did in class on 10/5 (there is also a copy of that Maple worksheet on the course homepage – `http://math.holycross.edu/~little/Applied9900/AM199.html` – for you to consult as needed).

A) Compute the Fourier coefficient B_n as a function of n for this initial temperature function, using numerical integration. First define the initial temperature function $f(x)$, then use:

```
B := n -> evalf((1/5)*Int(f(x)*sin(n*Pi*x/10),x=0..10));
```

(Technical Maple note: The capital "I" on the integrate function is important here. Maple has capitalized versions of a number of its basic functions. These are "inert" functions that are not evaluated immediately. Here by placing the "inert" integrate command *inside* the call to `evalf`, you are saying: compute the definite integral using an approximate numerical technique – *don't try to evaluate the integral symbolically first.*)

B) Plot the partial sums

$$\sum_{n=1}^{N} B_n \sin\left(\frac{n\pi x}{10}\right)$$

of the Fourier series of f together with f itself for several increasing values of N, until you see good agreement in the plots. (Note: As you increase N, these plots will take more time for Maple to compute, because the evaluation of the B_n is computationally intensive. Be patient!)

C) When you find a k that gives good agreement with the initial temperature function $f(x)$ in part B, make the corresponding partial sum of the series expansion (3) for the solution $u(x,t)$, and plot that as a function of x,t. Describe how the temperature throughout the rod evolves over time. What happens as $t \to \infty$, and why?

II. In this problem, we will consider a different type of BVP for the 1D heat equation. Instead of having the ends of the rod maintained at zero temperature, suppose now that the ends of the rod are *insulated.* As we have seen, this corresponds to the problem:

$$\begin{cases} u_t = k u_{xx} \\ u(x,0) = f(x) \\ u_x(0,t) = 0 \\ u_x(L,t) = 0 \end{cases} \tag{4}$$

Let $k = 0.01, L = 10$, and $f(x) = 40e^{-10(x-3)^2} + 20e^{-20(x-8)^2}$ as in question I.

A) The separation of variables works exactly the same way as above here (because the boundary conditions have not been used yet), and we get the product solutions $u(x,t) = G(t)\phi(x)$ as in (2) above. Now, show using the boundary conditions $u_x(0,t) = 0, u_x(L,t) = 0$ that the corresponding eigenvalues and eigenfunctions are $\lambda = 0$, $\phi(x) = 1$ and $\lambda = \left(\frac{n\pi}{L}\right)^2$, $\phi = \cos\left(\frac{n\pi x}{L}\right)$ for all integers $n \geq 1$. (This is the same as 2.3.2 part c, from Problem Set 3.)

B) To follow what we did before in the case where the ends of the rod were maintained at zero temperature, now we have to try to expand the initial temperature function $f(x)$ in terms of the eigenfunctions $1, \cos\left(\frac{n\pi x}{L}\right)$:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right).$$

1) Do problem 2.3.6 from the text.
2) Show that the result from 1 (2.3.6) leads to the following formulas:

$$\begin{cases} A_0 = \frac{1}{L}\int_0^L f(x)\, dx \\ A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx \quad \text{if } n \geq 1 \end{cases}$$

C) Compute the Fourier coefficients A_0 and A_n as a function of n for this initial temperature function, using numerical integration. (Follow what you did in I, part A.)

D) Plot the partial sums

$$A_0 + \sum_{n=1}^{N} A_n \cos\left(\frac{n\pi x}{L}\right)$$

of the Fourier series of f together with f itself for several increasing values of N, until you see good agreement in the plots.

E) When you find a N that gives good agreement with the initial temperature function $f(x)$ in part D, make the corresponding partial sum of the series expansion for the solution $u(x,t)$ (you'll need to figure out what that should look like), and plot that as a function of x,t. Describe how the temperature throughout the rod evolves over time. What happens as $t \to \infty$, and why?

III. *(Note: Most of this problem will require "pencil and paper" work rather than Maple calculations. You will use Maple only for the final part E.)* The 2D Laplace Equation is $u_{xx} + u_{yy} = 0$. The solutions of this equation represent *equilibrium temperature distributions* in 2D heat conduction problems. Suppose we have a rectangular $L \times H$ thin metal plate in the shape of $\{(x,y) : 0 \le x \le L, 0 \le y \le H\}$. Suppose the two edges of the rectangle with $x = 0$ and $x = L$ are *insulated*, while the temperature along the edge $y = 0$ is maintained at 0, and the temperature at the point (x, H) on the edge $y = H$ is given by a given function $f(x)$.

A) Explain why the conditions above are equivalent to the BVP

$$\begin{cases} u_{xx} + u_{yy} = 0 & \\ u_x(0,y) = 0 & \text{for all } y,\ 0 \le y \le H \\ u_x(L,y) = 0 & \text{for all } y \\ u(x,0) = 0 & \\ u(x,H) = f(x) & \end{cases}$$

B) To solve a problem of this form, we will apply the same separation of variables idea. Consider a function $u(x,y) = h(x)\phi(y)$. Show that a function of this form satisfies Laplace's equation if and only if

$$\frac{d^2h}{dx^2} = -\lambda h \quad \text{and} \quad \frac{d^2\phi}{dy^2} = \lambda\phi$$

(This the same as 2.3.1, part c from Problem Set 3.)

C) Because of the form of these separated equations, we can use (2) in each variable.

1) Use the boundary conditions $u_x(0,y) = 0$ and $u_x(L,y)$ to show that $\lambda \ge 0$ is the only possibility, and that the x eigenfunctions are $h(x) = 1, \cos\left(\frac{n\pi x}{L}\right)$ with corresponding eigenvalues $\lambda = 0, \left(\frac{n\pi}{L}\right)^2$.

2) Now we will use the boundary condition $u(x,0) = 0$. If $\lambda = 0$, show that the corresponding function $\phi(y)$ must be $\phi(y) = C_0 y$ for some constant C_0. If $\lambda = \left(\frac{n\pi}{L}\right)^2$, $n \ge 1$, show that the only corresponding eigenfunctions $\phi(y)$ are $C_n \sinh\left(\frac{n\pi y}{L}\right)$ for some constant C_n. This gives product solutions $1 \cdot y$, and $\cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi x}{L}\right)$ satisfying the first three boundary conditions.

D) Taking a linear superposition of the functions found in part C gives an expansion of the form:

$$u(x,y) = C_0 y + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right) \tag{5}$$

Note that we have not used the last boundary condition $u(x,H) = f(x)$ yet! If we substitute $y = H$ into the expansion above, we get

$$u(x,H) = C_0 H + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi H}{L}\right)$$

By comparing this with the Fourier cosine expansion of $f(x)$ as in question II:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

explain how to determine the C_n in (5).

E) Suppose $L = 10$, $H = 5$, and $f(x) = x(10 - x)$. Determine the corresponding solution of Laplace's Equation, find a partial sum of the series that approximates the function f reasonably well, and plot using Maple.

Assignment

Individual write-ups due Tuesday, October 26.