Mathematics 173 – Applied Mathematics 1
Review Sheet for Midterm 1
September 29, 1999

General Information

As announced in the course syllabus, the first midterm exam will be a closed book exam given in class on Thursday, October 7. The exam will take the entire class period that day. The exam will have three or four questions similar to questions from the problem sets.

Topics to Be Covered

The exam will cover the material from class, starting from the beginning of the semester through class on Thursday, September 30. This includes Chapter 1 and Chapter 2 sections 1,2,3 in the text, plus the extra material on vector calculus – see below. Specifically, this means (not in the order we talked about them in class):

1) Vector Calculus – line integrals, Green's Theorem, and the "alternate form", surface integrals, the Divergence Theorem. Know the precise statements of the two theorems and how to compute both sides in the equalities. (You should know the standard parametrizations for curves like lines, graphs ($y = f(x), x = g(y)$), circles, and ellipses, and surfaces like planes and spheres. They will not be given. Also be aware that it may "pay" to use polar, cylindrical, or spherical coordinates to evaluate some of the double and triple integrals that come up.)
2) The 1D, 2D, and 3D heat equations. Know the derivation of these equations from the conservation of energy principle as we saw in class for the 1D case and on Discussion 2 for the 2D and 3D cases. (Note: The 2D and 3D cases use a number of ideas from 1 above). Know how to verify that given functions are solutions.
3) Boundary conditions – know how to express various boundary conditions for the 1D heat equation such as
   a) "the ends of the rod are maintained at constant temperature"
   b) "the ends of the rod are insulated"
   c) "the heat flux at the ends equals ... "

   as equations.
4) Equilibrium solutions of boundary value problems for the heat equation (know how to derive them, and be able to interpret what they mean)
5) Laplace's Equation for equilibrium solutions of the 2D heat equation. Know the form of the Laplacian in rectangular and polar coordinates and how to use them. (But I will not ask you for the derivation of the polar coordinate form.)
6) Separation of variables for the 1D Heat Equation (ends maintained at zero temperature), solutions of the ordinary differential equation $\frac{d^2\phi}{dx^2} = -\lambda\phi$. Superposition of solutions.

Review Session

I would be happy to hold an evening review session before the exam next week. Is Tuesday evening good?

Some Review Problems

I. Let $\mathbf{F} = (x^2 - y^2, xy)$. Let D be the region in the plane bounded by the line $y = x$ and the parabola $y = x^2$, and let α be the positively oriented boundary of D. Show that the conclusion of Green's Theorem is valid by computing both sides of the equation.

II. Show using the "alternate form" of Green's Theorem that if f, g are any two functions with continuous second order partial derivatives on the region D bounded by a simple closed curve α, then

$$\int_\alpha f\nabla g \cdot \mathbf{N}\; ds = \iint_D f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g\; dA$$

(Here $\mathbf{N}$ is the outward unit normal to α.)

III. Let $\mathbf{F} = (y^2, x + y^2 + z, x - z)$, let Φ be the surface formed by the upper hemisphere of the sphere of radius 1 centered at (0,0,0), and the disk in the xy-plane inside $x^2 + y^2 = 1$. Say why the Divergence Theorem applies in this case, and compute both sides in the equality, showing they are the same.

IV. ("Convection") Assume we have a long thin wire or rod with constant thermal properties (c, ρ, K_0 all constant). Assume the wire is *not insulated along its lateral surface*, so that the rate of change (per unit time) of heat energy per unit surface area along the lateral surface is $C(u(x,t) - T)$, where T is the temperature of the medium surrounding the wire, and C is a positive constant.

A) Show that the heat equation in this situation takes the form

$$c\rho u_t = K_0 u_{xx} + Q - \frac{CP}{A}(u - T)$$

(A is the cross-section area of the wire; P is the perimeter of a cross-section.) Hint: Proceed as in our derivation of the 1D heat equation from the first day of class.

B) Suppose the surrounding temperature is $T = 0$, there are heat sources in the wire, and that the temperature in the rod is uniform ($u(x,t) = u(t)$, independent of x.) Determine $u(t)$, if $u(0) = u_0$ is given.

From the text: 1.4.1, 1.4.2, 1.4.6, 1.4.7, 1.5.9 (Assume constant thermal properties, assume the temperature is radially symmetric, and that there are no heat sources. Use polar coordinates!), 1.5.12, 1.5.13, 1.5.14 (Extra Credit problem), 2.3.2