Mathematics 174 – Applied Mathematics 2
Lab/Demo Day on Bessel Functions
February 15, 2000

Background

Last time, we introduced the Bessel functions of the first kind via the series expansions:

$$J_m(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(m+\ell)!}\left(\frac{z}{2}\right)^{2\ell+m} \tag{1}$$

Using the Ratio Test, we saw in discussion that the series for $J_m(z)$ in (1) converges for all $z \in \mathbf{R}$, so it gives a well-defined function on $\mathbf{R}$. (Actually, it also coverges for *complex* z as well, but that is another story!) At this point in your acquaintance with them, though, Bessel functions probably seem like "magic" since apart from the name $J_m(z)$ we have only the description (1). Today, we want to get more familiar with these functions and their properties using Maple to generate plots, work with some of the formulas, etc.

Note: This is not a conventional lab day leading to a report you will hand in. Instead, you should think of the worksheet you generate today (especially the graphs, and comments you add to explain things to yourself when you consult this in the future) *as your class notes for the day.* You might want to print out a copy and insert it in your notebook at this point.

Maple Details

The Bessel functions $J_m(x)$ are built-in in Maple. The Maple notation for this function is `BesselJ(m,x)`. You can think of `BesselJ(m,x)` as a function of two variables – the first (namely the m) is the *order*, the second is the actual *argument*, x. Since these functions are built-in, you can use them in a Maple session just like the elementary functions *sin, cos, tan, exp, ln, arctan* and so forth. For instance you can operate on Bessel functions with other Maple commands, you can plot them or functions made using them and other functions, etc.

Demonstrations, part 1

A) You can recover the series expansion (1) for $J_m(x)$ in the case that m is a positive integer using the `taylor` command in Maple. Try this command:

```
taylor(BesselJ(2,x),x=0,11);
```

The output is the terms of degree < 11 in the Taylor series of $J_2(x)$ at $x = 0$. Does this agree with (1) above? Check using Maple's `sum` command to write out the same terms as in (1) with $m = 2$.

B) Plot $J_0(x), J_1(x), J_2(x), J_3(x)$ together on the same set of axes, with $x \in [0, 10]$. Which function is which and how can you tell? Note: In (1), the term of lowest degree (in x)

in the series for $J_m(x)$ is x^m. What does that tell you about the shape of the graph $y = J_m(x)$ for x near zero?

C) The Bessel functions and their derivatives satisfy a plethora of different relationships. A few of them are given in the problem set for this week (questions D,E,H). There are *many* more as well. For example, try entering the command

```
diff(BesselJ(n,x),x);
```

to compute $J_n'(x)$. Note that the result is expressed in terms of other Bessel functions. How does this formula relate to the formulas given in problem D on the problem set? Next, what happens if you compute the second derivative, $J_n''(x)$ using

```
diff(BesselJ(n,x),x$2);
```

How does the result here compare with the relation between $J_n(x)$, $J_n'(x)$, $J_n''(x)$ given by Bessel's equation of order n? (It may help to `simplify` your results.)

D) As discussed in the problem set (question F), the Bessel function of order m can be defined even if m is not an integer if we use the *gamma function*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

to replace the factor $(m + \ell)!$ in the denominator of (1) by $\Gamma(m + \ell + 1)$:

$$J_m(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\, \Gamma(m+\ell+1)} \left(\frac{z}{2}\right)^{2\ell+m} \tag{2}$$

Plot several different $J_m(x)$ for different m that are not integers. What can you say if $m > 1$? How do these functions compare with the $J_m(x)$ if m is an integer. Compare the plots of $J_{1/2}(x)$ and $J_{-1/2}$ with the plots of the functions defined by the formulas given in question H of the problem set.

More Maple Details

Note that the graph $y = J_m(x)$ for $m \geq 0$ always looks more or less like a decaying oscillation. There is an intuitive physical explanation for this given in the text on pages 307-308. For us, the important consequence is that as $x \to \infty$, we get a *sequence of positive roots* of the equation $J_m(x) = 0$. These roots of the Bessel functions will be very important in our study of the vibrating circular drumhead. In addition to the `BesselJ` function, Maple also has a function called `BesselJZeros`. Try entering

```
evalf(BesselJZeros(3,4));
```

The result is a decimal approximation to the 4th positive zero of $J_3(x)$. (Note: "positive" means *strictly positive* here.) Compare with the plot of $y = J_3(x)$ to check this. If you leave out the `evalf` in the command above, Maple will just give you the input back again, since that is an *exact* way to specify the number in question (!) In general `BesselJZeros(m,n);` gives the nth positive root of $J_m(x) = 0$.

Demonstrations, part 2

E) By plotting and/or using `BesselJZeros`, examine the locations of the roots of $J_m(x) = 0$ and the locations of the roots of $J_{m+1}(x) = 0$. (Be sure you try several m's.) What general pattern do you notice? Is the same pattern apparently true for the roots of $J_m(x) = 0$ and $J_{m'}(x) = 0$ if $m \neq m'$ are any two different orders? Explain. You may assume m, m' are positive integers here.

F) (*"The punchline"*) We introduced the Bessel equation of order m:

$$x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - m^2) f = 0$$

because we wanted to study the eigenvalues and eigenfunctions for the boundary value problem:

$$(2) \qquad \begin{cases} \frac{d}{dr}\left(r \frac{df}{dr}\right) - \frac{m^2}{r} f = -\lambda r f \\ f(a) = 0 \\ |f(0)| < \infty \end{cases}$$

We let

$$L(f) = \frac{d}{dr}\left(r \frac{df}{dr}\right) - \frac{m^2}{r} f$$

denote the differential operator in (2). Using the Bessel function zeroes, we can construct the functions

```
f := BesselJ(m,BesselJZeros(m,n)*r/a);
```

Compute $L(f)$ using Maple, and `simplify`. What does the result tell you?

G) Take $m = 4$, $n = 1, 2, 3, 4, 5$ (one at a time of course), $a = 2$ and plot the functions f from part F on the interval $r \in [0, 2]$. Do these functions satisfy the properties we know for eigenfunctions of Sturm-Liouville boundary value problems? Explain.

H) What are the eigenvalues and eigenfunctions of the general problem (2). Give a complete description by generalizing what you did in parts F and G.