Mathematics 173 – Applied Mathematics 1
Problem Set 2 – Line and Surface Integrals
September 14, 1999

I. Evaluate the line integrals of each of the following vector fields along the indicated curves, and answer the accompanying questions.

A) $\mathbf{F} = (x, x+y)$, $\alpha(t) = (t - t^2, t^2 + t^3)$, $t \in [0,1]$. *Answer: 139/60.*
B) $\mathbf{F} = (y^2, 2xy)$, α the path from $(-1,2)$ to $(1,5)$ made up of two straight line segments, the first parallel to the x-axis, the second parallel to the y-axis.
C) $\mathbf{F} =$ same vector field as in part B, but now α the line segment directly from $(-1,2)$ to $(1,5)$.
D) Your answers for B and C should be equal. Explain why this is true. *Hint:* Is $\mathbf{F}$ a gradient vector field?
E) The vector field

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$

α the circle of radius 1 centered at $(0,0)$ in $\mathbf{R}^2$, traversed counterclockwise.

II. Use Green's Theorem to evaluate each of the following line integrals around the specified simple closed curve, and answer the other questions.

A) $\mathbf{F} = (\sin(\pi xy), \cos(\pi xy))$, α the positively-oriented boundary of the square $[0,1] \times [0,1]$. *Answer:* $-\frac{2+\pi}{\pi}$.
B) $\mathbf{F} = (y^2, xy)$, α the ellipse with equation $x^2/9 + y^2/4 = 1$.
C) Compute

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

for the vector field $\mathbf{F}$ and the region D inside α from part E of question I. Your answer should be different from what you saw in that part. Is this a contradiction to Green's Theorem?? Explain.

III. The statement of Green's Theorem is also valid in more general situations. For example, consider the *circular ring* D in the plane between the two circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$. The positively oriented boundary of D consists of the the circle of radius 2 traversed counterclockwise, and the circle of radius 1 traversed clockwise. Let $\mathbf{F} = (P, Q)$ be a vector field whose component functions have continous first order partial derivatives on some open set in $\mathbf{R}^2$ containing the whole ring. Show that the line integral of $\mathbf{F}$ over the positively oriented boundary is equal to the double integral of $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ over the ring. *Hint: Consider the curve α_1 which consists of the line segment from $(1,0)$ to $(2,0)$, followed by the upper semicircle of radius 2 traversed counterclockwise, followed the line segment from $(-2,0)$ to $(-1,0)$, followed by the upper semicircle of radius 1 traversed clockwise. Similarly let α_2 be the curve consisting of the line segment from $(-1,0)$ to $(-2,0)$, followed* *by the lower semicircle of radius 2 traversed counterclockwise, followed by the line segment from* $(2,0)$ *to* $(1,0)$*, followed by the lower semicircle of radius 1 traversed clockwise.*

IV. For each of the following vector fields and surfaces, verify the Divergence Theorem by direct computation of the surface integral and the triple integral:

A) $\mathbf{F} = (z, y, x)$, S the sphere of radius 1 centered at $(0, 0, 0)$.

B) $\mathbf{F} = (x + y, y + z, x + z)$, S the surface of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$. (For the surface integral, you will need to parametrize each of the four faces of the tetrahedron separately, and add the corresponding surface integrals.) *Answer: 1/2 (both values).*

Assignment

Individual solutions, due Tuesday, September 21.