Mathematics 173 – Applied Mathematics 1
Exam 1 Solutions – October 14, 1999

I.

A) The boundary of D consists of two pieces. The first, $\alpha_1(t)$, is the arc of the unit circle $(\cos t, \sin t)$ for $t \in [\pi/4, 5\pi/4]$ traced counterclockwise. The second, $\alpha_2(t) = (t,t)$, the segment of the line $y = x$ for $t \in [-\sqrt{2}/2, \sqrt{2}/2]$. The line integral is:

$$\begin{aligned}
\int_\alpha \mathbf{F}\cdot\mathbf{T}\; ds &= \int_{\alpha_1} \mathbf{F}\cdot\mathbf{T}\; ds + \int_{\alpha_2} \mathbf{F}\cdot\mathbf{T}\; ds \\
&= \int_{\pi/4}^{5\pi/4} (\cos^2 t - \sin t, \cos t + \sin t)\cdot(-\sin t, \cos t)\; dt \\
&\quad + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (t^2 - t, 2t)\cdot(1,1)\; dt \\
&= \int_{\pi/4}^{5\pi/4} -\cos^2 t \sin t + 1 + \sin t \cos t\; dt + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} t^2 + t\; dt \\
&= [(\cos^3 t)/3 + t - (\cos^2 t)/2]_{\pi/4}^{5\pi/4} + [t^3/3 + t^2/2]_{-\sqrt{2}/2}^{\sqrt{2}/2} \\
&= \frac{1}{3}(-\sqrt{2}/2)^3 + 5\pi/4 - \frac{1}{2}(-\sqrt{2}/2)^2 + \frac{1}{3}(\sqrt{2}/2)^3 + \frac{1}{2}(\sqrt{2}/2)^2 \\
&\quad - \frac{1}{3}(\sqrt{2}/2)^3 - \pi/4 + \frac{1}{2}(\sqrt{2}/2)^2 - \frac{1}{3}(-\sqrt{2}/2)^3 - \frac{1}{2}(-\sqrt{2}/2)^2 \\
&= \pi
\end{aligned}$$

B) First, since $\mathbf{F} = (x^2 - y, x + y) = (P, Q)$, we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

For the double integral over D, if rectangular coordinates are used, then the region must be split into two portions, since the equations of the upper and lower boundaries of the region are different for $x \in [-1, -\sqrt{2}/2]$ and for $x \in [-\sqrt{2}/2, \sqrt{2}/2]$. In the first range, the circle forms the upper and lower boundary; in the second, the line is the lower boundary. The double integral is the following sum of iterated integrals:

$$\int_{-1}^{-\sqrt{2}/2} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\; dy\; dx + \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{x}^{\sqrt{1-x^2}} 2\; dy\; dx$$

It is simpler to use polar coordinates, since the region of integration is $D = \{(r,\theta) : 0 \le r \le 1, \pi/4 \le \theta \le 5\pi/4\}$. We have

$$\iint_D 2\; dA = \int_{\pi/4}^{5\pi/4} \int_0^1 2r\; dr\; d\theta = \pi.$$

(This can also be seen since $\iint_D 2\, dA$ equals twice the area of D, which is $2 \cdot \pi/2 = \pi$.)

III.

A) To verify that $u(x,t) = e^{-kt}\sin x + \cos(x/\sqrt{k})$ is a solution of this boundary value problem, we may simply compute as follows:

$$\begin{aligned} u_t &= -ke^{-kt}\sin x \\ u_{xx} &= -e^{-kt}\sin x - \frac{1}{k}\cos(x/\sqrt{k}) \end{aligned}.$$

Hence,

$$\begin{aligned} ku_{xx} + \cos(x/\sqrt{k}) &= k[-e^{-kt}\sin x - \frac{1}{k}\cos(x/\sqrt{k})] + \cos(x/\sqrt{k}) \\ &= -ke^{-kt}\sin x - \cos(x/\sqrt{k}) + \cos(x/\sqrt{k}) \\ &= -ke^{-kt} = u_t \end{aligned}.$$

Then

$$\begin{aligned} u(x,0) &= e^0\sin(x) + \cos(x/\sqrt{k}) = \sin(x) + \cos(x/\sqrt{k}) \\ u(0,t) &= e^{-kt}\sin(0) + \cos(0) = 1 \\ u(\pi,t) &= e^{-kt}\sin(\pi) + \cos(\pi/\sqrt{k}) = \cos(\pi/\sqrt{k}) \end{aligned}$$

B) An equilibrium solution is one where $u_t \equiv 0$, so $u = u(x)$. In this case, the PDE $u_t = ku_{xx} + \cos(x/\sqrt{k})$ becomes the ordinary differential equation:

$$\frac{d^2u}{dx^2} = -\frac{1}{k}\cos(x/\sqrt{k}).$$

This can be solved for u by integrating twice with respect to x:

$$\begin{aligned} \frac{du}{dx} &= \int \frac{d^2u}{dx^2}\ dx \\ &= \int -\frac{1}{k}\cos(x/\sqrt{k})\ dx \\ &= -\frac{\sqrt{k}}{k}\sin(x/\sqrt{k}) + c_1 \end{aligned}$$

Then,

$$\begin{aligned} u &= \int -\frac{\sqrt{k}}{k}\sin(x/\sqrt{k}) + c_1\ dx \\ &= \cos(x/\sqrt{k}) + c_1x + c_2 \end{aligned}$$

where c_1, c_2 are constants. Using the boundary conditions, at $x = 0$, $u(0) = 0 \Rightarrow 0 = \cos(0) + c_2$. Hence $c_2 = -1$. Then at $x = \pi$, $u(\pi) = 0$ implies $0 = \cos(\pi/\sqrt{k}) + c_1\pi - 1$. Solving for c_1, we get

$$c_1 = \frac{1 - \cos(\pi/\sqrt{k})}{\pi}.$$

IV.

A) By the linearity of differentiation and the assumption that $(u_n)_t = k(u_n)_{xx}$ for all n, if $u(x,t) = \sum_n c_n u_n(x,t)$, then

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}\left(\sum_n c_n u_n(x,t)\right) \\ &= \sum_n c_n (u_n)_t \\ &= \sum_n c_n k (u_n)_{xx} \\ &= k\frac{\partial^2}{\partial x^2}\left(\sum_n c_n u_n\right) \\ &= k u_{xx} \end{aligned}$$

Similarly, if each u_n satisfies $u_n(0,t) = 0 = u_n(L,t)$, then $u(0,t) = \sum_n c_n u_n(0,t) = \sum_n c_n \cdot 0 = 0$ and $u(L,t) = \sum_n c_n u_n(L,t) = \sum_n c_n \cdot 0 = 0$.

B) By "separation of variables plus superposition" recipe, if

$$\begin{aligned} u(x,0) &= \sin\left(\frac{3\pi x}{L}\right) - 5\sin\left(\frac{7\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

then the solution of this boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} d_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$$

The d_n are found by the formula

$$\begin{aligned} d_n &= \frac{2}{L}\int_0^L \left(\sin\left(\frac{3\pi x}{L}\right) - 5\sin\left(\frac{7\pi x}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{2}{L}\int_0^L \sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &- 5\frac{2}{L}\int_0^L \sin\left(\frac{7\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \begin{cases} 1 & \text{if } n = 3 \\ -5 & \text{if } n = 7 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by the orthogonality relation. This shows

$$u(x,t) = u_3(x,t) - 5u_7(x,t) = e^{-k\left(\frac{3\pi}{L}\right)^2 t}\sin\left(\frac{3\pi x}{L}\right) - 5e^{-k\left(\frac{7\pi}{L}\right)^2 t}\sin\left(\frac{7\pi x}{L}\right)$$