Mathematics 173 – Applied Mathematics 1
Discussion 3 – The Vibrating Drumhead
November 30, 1999

Background

The *2D wave equation*

$$u_{tt} = c^2(u_{xx} + u_{yy}) \tag{1}$$

describes motions of a vibrating membrane such as a drumhead. We will consider the case where the membrane's rest position is the rectangle $[0, L] \times [0, H]$ in the xy-plane. The function $u(x, y, t)$ is the vertical displacement of the point $(x, y) \in [0, L] \times [0, H]$ at time t. Before Thanksgiving, we saw by the separation of variables technique that the "completely separated" product solutions of (1) satisfying the boundary conditions

$$\begin{cases} u(0, y, t) = 0 & \text{all } y, t \\ u(L, y, t) = 0 & \text{all } y, t \\ u(x, 0, t) = 0 & \text{all } x, t \\ u(x, H, t) = 0 & \text{all } x, t \end{cases}$$

look like:

$$u(x, y, t) = \Big(A_{mn}\cos(\sqrt{\lambda_{mn}} \cdot ct) + B_{mn}\sin(\sqrt{\lambda_{mn}} \cdot ct)\Big)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)$$

where m, n runs over the set of all pairs of integers ≥ 1, and

$$\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

These are the *normal modes of vibration* of the drumhead. Today we want to study these and understand what the corresponding vibration patterns look like.

Discussion Questions

A) First consider the case $n = 1$, $m = 1$ at a fixed but arbitrary time t_0. How many critical points does each $u(x, y, t_0)$ have? Use the Second Derivative Test to classify each of the critical points in the interior of the drumhead as a local maximum, a local minimum, or a saddle point. You can ignore critical points of u on the boundary of the drumhead. Your answer should depend on what t_0 is – how does that work? By considering how the character of the critical points changes with t, describe the motion of the drumhead in words. There are points (x, y) where $u(x, y, t)$ does not change with t; these are called "nodal lines"? Where are they?

B) Now do the same for the case $n = 2$, $m = 1$. Find the critical points for this u, determine how their type (local maximum, local minimum, or saddle point) varies with t, and describe this motion, comparing it with your answer from part A. What differences are there? Apart from the difference in the numbers of critical points, what is the effect of the factor of $\sqrt{\lambda_{mn}}$ in the time-dependent factor?

C) What is the pattern here for general $m, n \geq 1$. What does the m, n normal mode of vibration look like? How many critical points are there? How do they change with t? Where are the nodal lines? How does the period of the vertical oscillations depend on m, n? Note: Maple can be used to good effect if you would like to plot, or even animate these!!

Assignment

Group writeups due Monday, December 8 – the last day of classes of the semester.