Mathematics 173 – Applied Mathematics 1
Discussion 2 – The 3D (and 2D) Heat Equations
September 21, 1999

Background

Last Thursday, we saw that for a 3D solid body S with specific heat c and mass density ρ, if we consider the change of heat energy with respect to time in every subregion $R \subseteq S$, then we have the following "Conservation of (heat) Energy" equation relating the temperature function u, and the heat flux vector field, $\mathbf{F}$:

$$(CE) \qquad \iiint_R c\rho \frac{\partial u}{\partial t}\, dV = -\iint_\Phi \mathbf{F}\cdot\mathbf{N}\, dA + \iiint_R Q\, dV.$$

Here Φ is any parametrization of the boundary surface of R, and $\mathbf{N}$ is the outward unit normal vector to the boundary. Note the minus sign on the surface integral term. Since $\mathbf{N}$ points outward, a positive value for $\iint_\Phi \mathbf{F}\cdot\mathbf{N}\, dA$ should indicate that heat energy is *leaving* R, and conversely a negative value for $\iint_\Phi \mathbf{F}\cdot\mathbf{N}\, dA$ should indicate that heat energy is flowing across the boundary into R. This explains that sign.

Applying the Divergence Theorem to the first term on the right, we get:

$$\iiint_R c\rho \frac{\partial u}{\partial t}\, dV = -\iiint_R div(\mathbf{F})\, dV + \iiint_R Q\, dV.$$

Hence we can combine all terms on one side of the equation as follows:

$$\iiint_R c\rho \frac{\partial u}{\partial t} + div(\mathbf{F}) - Q\, dV = 0$$

We saw that since this equation is satisfied on *every* subregion $R \subseteq S$, as long as the integrand is continuous (which is a reasonable assumption), then

$$(1) \qquad c\rho \frac{\partial u}{\partial t} = -div(\mathbf{F}) + Q$$

identically on S.

Today, we want to use this to develop the form of the heat equation in 3 (and 2) space dimensions, and consider some related questions.

Discussion Questions

A) Recall that heat energy flows from regions of higher temperature to regions of lower temperature. The analog of Fourier's Law that is valid in this situation is: *The heat flux vector* $\mathbf{F}$ *points in the direction in which the temperature* u *is decreasing the fastest.* Express this statement as an equation. (What vector we know will always point in the direction in which a scalar function is *decreasing the fastest*?) You will need to introduce a scalar *thermal conductivity function* $K_0(x, y, z, t)$ here as a proportionality factor as we did previously.

B) Continuing from part A, substitute your expression for $\mathbf{F}$ into (1) to derive the 3D Heat Equation. Show that if c, ρ, K_0 are all constant, then the equation can be rewritten in the form

$$(2) \qquad \frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + Q.$$

C) The expression $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ in (2) is called the *Laplacian* of the function u. Show that the Laplacian is the divergence of ∇u (the gradient vector field of u). Formally, the Laplacian is $\nabla \cdot \nabla u$, so it is often written as $\nabla^2 u$.

D) Give the analog of the equation (CE) above for a thin 2D plate of material in the shape of a region $S \subset \mathbf{R}^2$, and show using our "alternate form of Green's Theorem" that the 2D Heat Equation for a thin plate with constant c, ρ, K_0 can be written in a completely analogous way to (2):

$$(3) \qquad \frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + Q.$$

E) Assume the heat source term $Q = 0$ in equation (3). Show in (3) that any steady-state solution $u = u(x, y)$ (independent of t) satisfies

$$(4), \qquad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

a relation called *Laplace's Equation.* (Note: this is also a PDE for the function $u(x, y)$.) Show that $u = x^3 - 3xy^2$, $u = e^x \cos(y)$, $u = \frac{x}{x^2+y^2}$ are all solutions of Laplace's Equation. Also, find a non-steady state solution of (3) that tends to $u = x^3 - 3xy^2$ as $t \to \infty$. What does this say about steady state solutions of the 2D Heat Equation (with $Q = 0$) as compared to the steady state solutions for the 1D equation (the case of a long, thin rod)?

F) Consider the 2D and 3D Heat Equations from (2) and (3). To express a boundary condition saying that u has known values on the boundary of S (shorthand notation: ∂S), we could just give an equation $u = T$ for all points $\in \partial S$ and all t. How would you express the condition that *the boundary of S is insulated* as an equation? (What does insulation imply about the heat flux vector at points on the boundary?)

Assignment

Group write-ups due Tuesday, September 28