Mathematics 173 – Applied Mathematics 1
Discussion 1 – The Heat Equation
September 2, 1999

Background

Recall that last time, we gave a derivation of the Heat Equation for a long, thin rod or wire:

$$(HE) \qquad c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q(x,t),$$

where:

- $u = u(x,t)$ is the *temperature* at location x in the wire, at time t,
- $c(x)$ is the *specific heat* of the material in the wire at location x (the amount of heat energy needed to raise a unit mass one degree),
- $\rho(x)$ is the *mass density* of the material at location x,
- $K_0(x,t)$ is the *thermal conductivity* of the material,
- we assumed the lateral surface of the rod or wire is insulated so not heat is lost or gained through that surface.

This is an example of a *partial differential equation* (PDE). We think of c, ρ, K_0, Q as *known functions* (known once we know the particular properties of the rod we are studying). The temperature function u is the unknown, and (HE) gives a relation between partial derivatives of u, and various known functions.

An important special case is the case where c, ρ, K_0 are all *constant*. That is, the thermal properties do not depend on the position within the wire, or on time – we say the rod is *uniform*. If we have a uniform rod, containing no heat sources (so $Q(x,t) = 0$ all x, t), then the heat equation reduces to

$$(HE') \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where $k = \frac{K_0}{c\rho}$ is also constant.

Discussion Questions

A) Show that each of the following functions is a *solution* of the heat equation for a uniform rod with no heat sources – that its partial derivatives satisfy the relationship (HE'). Also answer each of the questions connected with the specific solutions.
   1) $u_1(x,t) = ax + b$ (a,b constant). What is "special" about this type of solution? (What happens to the temperature as t changes?)
   2) $u_2(x,t) = x(x^2 + 6kt)$.

3) $u_3(x,t) = e^{-kt}\sin(x)$ and $u_4(x,t) = e^{-kt}\cos(x)$. Suppose the wire extends from $x = 0$ to $x = L = \pi$. Is one of these solutions more reasonable physically than the other? (Recall that $u = 0$ is the lowest possible temperature.)

4)
$$u_5(x,t) = \frac{1}{\sqrt{t}} e^{\frac{-x^2}{4kt}}$$

Note that u_5 is not defined at $t = 0$. Plot $u_5(x,t)$ for $k = 1$ and several different t (at least $t = 1/4, 1/2, 1, 2$) and describe what is happening as t increases. What do you think the graph of $\lim_{t\to 0^+} u_5(x,t)$ looks like?

B) One possible question we could ask is: Given a uniform rod with known c, ρ, K_0, and no heat sources, if the *initial temperature* $u(x,0)$ is known for all x, does knowing that $u(x,t)$ satisfies the equation (HE') determine the value of $u(x,t)$ for all $t > 0$? Think about this on physical grounds. Should the answer be yes or no? Take into account the following two different scenarios:

scenario 1: The ends of the rod are maintained at constant temperature $T = 0$.
scenario 2: The ends of the rod are *insulated* so that no heat energy passes through the ends. (Note the lateral surface of the rod is insulated too. In this scenario, there is no transfer of heat energy between the rod and the "outside world" whatsoever).

Describe what you think should happen in each of these scenarios if $u(x,0) = f(x)$ is some known function. Do you think you get the same temperature function $u(x,t)$ for all $t > 0$ in these cases?

C) Express the two scenarios above in the form of *equations* involving $u(x,t)$ and its derivatives, assuming the ends of the rod are $x = 0$ and $x = L$. These are examples of what are known as *boundary conditions.* (Hint: For scenario 2, you probably want to think about Fourier's Law.)

D) Another form of boundary condition occurs if we assume that the heat flux at one or both boundaries is *proportional to the difference between the temperature of the rod at that end and the ambient temperature surrounding that end of the rod.* Express this form of boundary condition as a pair of equations if we have a rod in room maintained at a constant ambient temperature of T_a degrees.

Assignment

Group write-ups due in class on Thursday, September 9.