Mathematics 174 – Applied Mathematics 2
Problem Set 6 – An Analog Filter
April 6, 2000

In this problem set we will study a rudimentary analog filter – the RC circuit we discussed in class. Recall that for any input signal $f(x)$, where x represents *time* in this example, the voltage response of the system represented by the circuit is given by a solution $v(x) = \Gamma(f)(x)$ of the first order ODE:

$$\frac{dv}{dx} + \frac{1}{RC}v = \frac{f}{RC} \tag{1}$$

(where $R, C > 0$ are constants – the resistance of the resistor and the capacitance of the capacitor in the circuit).

A. We will assume that the response to a zero input $f \equiv 0$ is zero. (In physical terms, this means that there is no initial energy in the system before the input is supplied.) Show that in that case the response to a general input f can be written as

$$v(x) = \Gamma(f)(x) = \frac{1}{RC} e^{\frac{-x}{RC}} \int_{-\infty}^{x} f(s) e^{\frac{s}{RC}}\, ds \tag{2}$$

B. Explain why this system is *linear, time-invariant, and causal.*

C. Rewriting integral on the right hand side in (2) as a convolution, identify the *impulse response* $\gamma = \Gamma(\delta)$ of the system (the response to a delta function input at $x = 0$).

D. What is the response of this system to the Heaviside step function input:

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

E. Show that if $f(x)$ is any input signal, and $F(\omega) = \mathcal{F}[f](\omega)$ is its Fourier transform, then the Fourier transform G of the corresponding output $g = \Gamma(f)$ is given by

$$G(\omega) = \mathcal{F}[\Gamma(f)](\omega) = 2\pi F(\omega) H(\omega),$$

where H is the Fourier transform of $\gamma = \Gamma(\delta)$. The function $2\pi H(\omega)$ is called the *frequency response* of the system.

F. Show that the *frequency response* $H(\omega)$ of the RC circuit is:

$$H(\omega) = \frac{1}{1 + iRC\omega}$$

G. The function $|H(\omega)|$ is called the *magnitude response* of the system. Plot this as a function of ω, taking $R = 1, C = 5$. Also, determine the interval of ω values for which $|H(\omega)| \geq \frac{1}{\sqrt{2}}$.

The system here is a realizable *low-pass filter*, whose "pass-band" is taken (by convention) to be your interval from part G. Of course, a good low-pass filter should have a sharper pass-band cut-off, and should have $|H(\omega)|$ closer to 1 on more of its pass-band. One sub-area of signal processing studies methods for constructing such filters using more complicated circuits. The main tool is the form of the frequency response $H(\omega)$.

H. Plot the functions

$$|H(\omega)| = \frac{1}{\sqrt{1 + (5\omega)^{2N}}}$$

for $N = 2, 3, 4$. These magnitude response functions correspond to a family of realizable low-pass filters called *Butterworth filters*. Each can be realized by an electrical circuit like the RC circuit above, but with more components.

Assignment

Individual write-ups due Thursday, April 13.