Mathematics 174 – Applied Mathematics 2
Midterm Problem Set
February 24, 2000

Groundrules and Directions

This is an open-book, open-notes, open-library, open Swords 335, unlimited time assignment. If you use something you find in a book other than Haberman in one of your solutions, please footnote it. You must work independently, and you may not consult other faculty members in working on these problems. *When you have completed the problem set, please write a brief statement saying that you have abided by these ground rules, sign it, and turn it in with your solutions.*

In all parts of problems where solution of a BVP is asked for, give your answer in series form, *together with formulas for the coefficients.*

Don't kill trees unnecessarily for the Maple output you submit: delete irrelevant work and *resize your plots* before printing!!

Due: No later than 5:00 pm, Thursday, March 2.

Standing Assumptions: In all parts of both problems, m is an integer and $m \geq 0$.

I. For the 2D heat equation on a circular disk of radius a, if the outer edge is *insulated*, we are led to consider the problem

$$\begin{cases} u_t = k\nabla^2 u \\ u_r(a,\theta,t) = 0 \\ u(r,\theta,0) = \alpha(r,\theta) \end{cases}$$

A) If we look for product solutions $u(r,\theta,t) = f(r)g(\theta)h(t)$, what ODE's must h and g satisfy, and what boundary value problem for $f(r)$ results?
B) Show that for each Bessel function of the first kind $J_m(z)$ there is an infinite sequence of numbers $w_{mn} \geq 0$ where $J_m'(w_{mn}) = 0$. (Note: I'm looking for an actual proof here, given the facts we know about $J_m(z)$, its zeroes, etc. What theorem from 1-variable calculus guarantees the existence of places where the derivative of a function is zero?)
C) What are the eigenvalues and eigenfunctions of the boundary value problem for $f(r)$?
D) Solve the problem:

$$\begin{cases} u_t = k\nabla^2 u \\ u_r(a,\theta,t) = 0 \\ u(r,\theta,0) = r\sin(3\theta) \end{cases}$$

II. In this problem, you will study the eigenvalues and eigenfunctions of the Sturm-Liouville problem:

$$(1) \qquad \begin{cases} \frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r}f = -\lambda r f \\ f(a) = 0 \\ f(b) = 0 \end{cases} \qquad \text{where } a > 0$$

arising from problems on annular (ring-shaped) regions, and use them to solve several boundary value problems.

In class on 2/22, we saw that the change of variables $z^2 = \lambda r^2$ takes this to a new type of boundary value problem for the Bessel equation of order m:

$$(2) \qquad \begin{cases} z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2)f = 0 \\ f(\sqrt{\lambda}a) = 0 \\ f(\sqrt{\lambda}b) = 0 \end{cases}$$

(Note: the square roots are justified here since all eigenvalues satisfy $\lambda \geq 0$, as we will see shortly.)

The *general solution* of Bessel's equation of order m is a linear combination of the Bessel function of the first kind, $J_m(z)$, and the Bessel function of the second kind, which is called $Y_m(z)$:

$$f(z) = c_1 J_m(z) + c_2 Y_m(z)$$

For $m \geq 1$, $Y_m(z) \sim z^{-m}$ as $z \to 0$, so it is not defined for $z = 0$. (Similarly, $Y_0(z) \sim c\ln(z)$ for a constant c, so it is not defined for $z = 0$ either.) That is why we have not used (or needed) $Y_m(z)$ before – in all of our work on circular domains, we have wanted solutions f of the Bessel equation with $|f(0)| < \infty$. Now, however, our boundary conditions are at $a, b > 0$ and we are interested in functions on the interval $[a, b]$. Their behavior at $z = 0$ is not relevant!

A) Show that for each integer $m \geq 0$, we will get an infinite sequence of eigenvalues λ_{mn} of (1), $n = 1, 2, 3, \ldots$, and corresponding eigenfunctions $\phi_{mn}(r)$ determined uniquely up to a constant multiple.
B) Using the Rayleigh quotient for (1), show that all eigenvalues satisfy $\lambda_{mn} \geq 0$.
C) Is there a smallest eigenvalue overall? If so, for which m is the minimum achieved? If not, say why not.
D) Show that if $f(z) = c_1 J_m(z) + c_2 Y_m(z)$ is a nonzero function satisfying (2), then

$$(3) \qquad J_m(\sqrt{\lambda}a)Y_m(\sqrt{\lambda}b) - J_m(\sqrt{\lambda}b)Y_m(\sqrt{\lambda}a) = 0$$

E) The relation (3) in part D is actually *an equation for* λ. It cannot be solved exactly by any usual technique, but we can get approximations to solutions by graphical and numerical methods in Maple. Taking $a = 1$ and $b = 2$, by plotting, estimate the locations of the first 4 solutions of (3) for $m = 0, 1, 2$. Then use the `fsolve` command in Maple to generate accurate approximations of each of your roots. Notes: The Bessel function of the second kind of order m is called `BesselY(m,z)` in Maple. The command `fsolve(equation(var),var=c);` will use a numerical method similar to Newton's Method to solve the equation `equation(var) = 0` for the variable var, starting from the initial approximation $var = c$.
F) Once you have an approximation for λ_{mn}, how do you get an approximation to the corresponding eigenfunction $\phi_{mn}(r)$?

G) Studying the vibrations of an annular (ring-shaped) membrane with inner ($r = a$) and outer ($r = b$) edges fixed at zero displacement leads to consideration of the boundary value/initial value problems

$$(4) \qquad \begin{cases} u_{tt} = c^2 \nabla^2 u \\ u(a, \theta, t) = 0 \\ u(b, \theta, t) = 0 \\ u(r, \theta, 0) = \alpha(r, \theta) \\ u_t(r, \theta, 0) = \beta(r, \theta) \end{cases}$$

As always, the product solutions $u(r, \theta, t) = f(r)g(\theta)h(t)$ are called the *normal modes of vibration.* There are four types of normal modes ("cosine-cosine, cosine-sine, etc.") of (4) for each pair of integers (m, n) with $m \geq 0$, $n \geq 1$. Incorporating your work from parts A through F, use Maple to plot (approximations to) the "cosine-cosine" normal modes with $m = 0, 1, 2$, $n = 1, 2, 3, 4$ for a ring with $a = 1$ and $b = 2$. (Hand in plots with $t = \pi/4$; you can try animations too if you like!) Discuss the results, giving numbers and locations of nodal curves, etc.

H) Using your work from parts A through F again, solve the following BVP for the heat equation on an annular region as simply as possible:

$$\begin{cases} u_t = k \nabla^2 u \\ u(a, \theta, t) = 0 \\ u(b, \theta, t) = 0 \\ u(r, \theta, 0) = (r - a)(b - r) \sin(2\theta) \end{cases}$$