Mathematics 174 – Applied Mathematics 2
Final Problem Set
April 27, 2000

Ground Rules

This is an open book, open notes, open library, open Maple, open SW 335, unlimited time exam. If you find something of use in a library book, please footnote it (this is not necessary for topics covered in our course textbook, Haberman). You must work independently, and you may not consult other faculty members in working on these problems. *When you have completed the problem set, please write a brief statement saying that you have abided by these ground rules, sign it, and turn it in with your solutions.*

Due: No later than 5:00 pm, Friday, May 5.

Directions

Do problem I below, and any *two* of problems II, III, IV. (If you submit solutions for all four problems (or parts of them) I will give Extra Credit points.)

I. (40) In this problem, you will see another example of how Fourier Transforms can be applied to solve PDE. Consider the following problem for the *linearized Korteweg-deVries equation*

$$\begin{cases} u_t = u_{xxx} \\ u(x,0) = f(x) \end{cases} \tag{1}$$

(This is a simplified version of an equation used to model the motion of water waves in long, shallow channels like canals – $u(x,t)$ represents the vertical displacement at position x along the canal, at time t. For a long canal, and a wave that starts due to some disturbance of the water surface in the middle, there are essentially no boundary conditions, so we take $-\infty < x < \infty$.)

A) Take the Fourier Transform (with respect to x) of both sides of the PDE to find an ODE (with independent variable $= t$) satisfied by $U(\omega,t) = \mathcal{F}[u(x,t)]$ and solve it to find $U(\omega,t)$. Don't forget to include a "constant of integration" here, which will depend on ω. Take the inverse transform to give one way find a solution of (1).
B) Use the convolution theorem for the Fourier Transform to show that the solution $u(x,t)$ can be written as

$$u(x,t) = f(x) * G(x,t) = \int_{-\infty}^{\infty} f(s)G(x-s,t)\, ds$$

where the function G is given by

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(\omega^3 t - \omega x)\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos(\omega^3 t - \omega x)\, d\omega.$$

This is called the *fundamental solution* – it is the solution corresponding to $u(x,0) = \delta(x)$ (Dirac delta).

By a happy chance, there is a built-in function called `AiryAi` in Maple (with an efficient method for computing values) that is very closely related to this function:

$$G(x,t) = \frac{\text{AiryAi}\left(\frac{-x}{\sqrt[3]{3t}}\right)}{\sqrt[3]{3t}}$$

This is good because the integral representation above is *very inefficient* for evaluating $G(x,t)$ – it's similar to a very slowly-converging alternating series.

C) Use Maple to generate plots of the solution of (1) if

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

and describe what the solution $u(x,t)$ appears to be doing as t increases. Note: there are several different ways to think about computing the convolution formula from part B (either integrating directly, or by using the FFT in a suitable way to approximate). Explain how you are doing that step.

II. (30) Consider a Sturm-Liouville form differential equation:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0, \tag{2}$$

and assume $p(x) > 0$ for all $x \in [a,b]$ (as in a regular Sturm-Liouville eigenvalue problem). In this problem, we will study the *zeroes* of solutions of (2), and prove a result called the *Sturm Separation Theorem* – part B below. Let y_1, y_2 be any two solutions of (2). The *Wronskian* of y_1, y_2 is defined to be:

$$W[y_1,y_2](x) = \det\begin{pmatrix} y_1(x) & y_2(x) \\ \frac{dy_1}{dx}(x) & \frac{dy_2}{dx}(x) \end{pmatrix}$$

A) Show that $W[y_1,y_2](x) = C/p(x)$ for some constant C. (**Hint:** What is the derivative $\frac{d}{dx}W[y_1,y_2](x)$?) Use this to show that if the Wronskian of y_1 and y_2 is zero at any $c \in [a,b]$, then y_1 and y_2 are *linearly dependent*. You will need to use the fact that for all $c \in [a,b]$, and all $A, B \in \mathbf{R}$, there is a *unique* solution of (1), together with the initial conditions $y(c) = A$, $\frac{dy}{dx}(c) = B$.

B) Let y_1, y_2 be two linearly independent solutions of (2). Show that between any two consecutive zeroes of y_1 in $[a,b]$, there is precisely one zero of y_2 as follows: Assume $x_1 < x_2$ are the consecutive zeroes of y_1, but that y_2 is not zero on the interval $[x_1,x_2]$. Consider the function $\frac{y_1}{y_2}$ and apply the "most valuable theorem" (MVT) of calculus.

C) What does this result say about the zeroes of the Bessel functions $J_n(z)$ and $Y_n(z)$?

III. (30) In this problem, you will see a proof of the *Uncertainty Principle*, which implies (among other things) that signals *cannot* be simultaneously arbitrarily sharply time-limited and frequency-limited. The famous Heisenberg uncertainty principle from quantum mechanics is also derived from this fact about Fourier analysis. Throughout this problem, we will use the following notation (similar to our inner-products from Sturm-Liouville theory): Let $f(x), g(x)$ be "nice" signals (i.e. piecewise smooth, and in $L^1(\mathbf{R})$). We write

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx \tag{3}$$

where $\overline{g(x)}$ is the complex conjugate of $g(x)$ (i.e. if $g(x) = u(x) + iv(x)$, then $\overline{g(x)} = u(x) - iv(x)$). We also write $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2\, dx}$ (called the L^2-norm of f).

A) First, we need an important fact called the *Plancherel identity.* Show using the Fourier Inversion Theorem that

$$\langle \mathcal{F}[f], \mathcal{F}[g]\rangle = \frac{1}{2\pi}\langle f, g\rangle,$$

where the left-hand side is the inner product from (3) above, but for functions of ω. From this deduce the Plancherel identity:

$$\frac{1}{2\pi}\|f\|^2 = \|\mathcal{F}[f]\|^2.$$

B) We assume all signals are "centered" at $x = 0$. The statement of the Uncertainty Principle will involve a measure of the "effective width" of a signal and its Fourier transform. For the effective widths we will use the following definitions. The effective width of f is

$$\begin{aligned} W_f &= \frac{\|xf(x)\|}{\|f(x)\|} \\ &= \left[\frac{1}{\|f\|^2}\int_{-\infty}^{\infty} x^2|f(x)|^2\, dx\right]^{\frac{1}{2}} \end{aligned}$$

(this is related to a *variance* as defined in statistics; if that doesn't mean anything to you, ignore it!!) Similarly, we define the effective width of the Fourier transform to be

$$\begin{aligned} W_F &= \frac{\|\omega\mathcal{F}[f](\omega)\|}{\|\mathcal{F}[f](\omega)\|} \\ &= \left[\frac{2\pi}{\|f\|^2}\int_{-\infty}^{\infty} \omega^2|\mathcal{F}[f](\omega)|^2\, d\omega\right]^{\frac{1}{2}} \end{aligned}$$

Compute the effective width W_f for $f(x) = e^{-ax^2}$ (scaled Gaussian), and W_F for the Fourier transform of the Gaussian. What is the product $W_f W_F$ in this case?

C) The Uncertainty Principle is the statement:

$$W_f W_F \geq \frac{1}{2}$$

for all *real* signals f. (Hence, in particular, W_f and W_F cannot both be small!) Prove this statement by 1) recognizing that W_F is equal to $\frac{\|f'\|}{\|f\|}$, 2) applying the Cauchy-Schwarz inequality:

$$\|xf(x)\|\|f'(x)\| \geq \left| \int_{-\infty}^{\infty} xf(x)f'(x)\, dx \right|,$$

then 3) working with the integral on the right above. (You may assume $x(f(x))^2 \to 0$ as $x \to \pm\infty$.)

IV. (30) In this problem, you will work through most of the steps of a proof of the *Shannon-Nyquist Sampling Theorem*, which is the following statement:

Sampling Theorem. *Let f be piecewise smooth and in $L^1(\mathbf{R})$. Assume that $\mathcal{F}(\omega) = 0$ for all $|\omega| \geq L$. Then*

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \frac{\sin(Lx - n\pi)}{Lx - n\pi}. \tag{4}$$

The meaning of this is that if f is *frequency limited* (that is, $\mathcal{F}[f](\omega) = 0$ for all $|\omega| \geq L$), then we can reconstruct f *exactly* (in principle) from sampled values $f\left(\frac{n\pi}{L}\right)$, provided that the sampling interval is $\frac{\pi}{L} = \frac{2\pi}{2L}$. Any sampling interval $\Delta x < \frac{2\pi}{2L}$ would also work.

A) By the Fourier Inversion Theorem,

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}[f](\omega)e^{-i\omega x}\, d\omega = \int_{-L}^{L} \mathcal{F}[f](\omega)e^{-i\omega x}\, d\omega$$

Show that the Fourier *series* of $F(\omega) = \mathcal{F}[f](\omega)$ on $[-L, L]$ is:

$$F(\omega) \sim \frac{1}{2L} \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{\frac{in\pi\omega}{L}}$$

(see section 3.6 in Haberman, or the class notes, for the notation of the complex form of the Fourier series).

B) Let $H_L(\omega)$ be the periodic function defined by the Fourier series in part A. Explain why

$$F(\omega) \text{ ``='' } H_L(\omega)\Pi_L(\omega),$$

so

$$F(\omega) \text{ ``='' } \tfrac{1}{2L} \textstyle\sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{\frac{in\pi\omega}{L}} \Pi_L(\omega)$$

where $\Pi_L(\omega) = 1$ if $|\omega| \leq L$ and $= 0$ otherwise. Why are several equalities here written with quotation marks? (These are several technical details that are being overlooked for simplicity!)

C) Derive equation (4) by taking the Inverse Fourier Transform of both sides of the last equation.

Have a wonderful summer!