Mathematics 174 – Applied Mathematics 2
Discussion 6 – The Discrete Fourier Transform
April 13, 2000

Background

Let $\{f_j\}_{j=0}^{N-1}$ be a (finite) sequence of complex numbers. The N-point discrete Fourier transform (DFT) of $\{f_j\}$ is the sequence F_k, $k \in \mathbf{Z}$, defined by

$$F_k = \frac{1}{2\pi} \sum_{j=0}^{N-1} f_j \; e^{\frac{2\pi i (jk)}{N}}$$

If the f_j are samples of an analog signal f on the interval $[0, \Omega]$: $f_j = f\left(\frac{j\Omega}{N}\right)$, then the DFT sequence gives *approximate values for samples of the Fourier transform of f* as follows:

$$F_k \doteq \frac{N}{\Omega} \mathcal{F}[f] \left(\frac{2\pi k}{\Omega}\right). \tag{1}$$

The complex numbers $z_\ell = e^{\frac{2\pi i \ell}{N}}$ are all Nth roots of unity: For all $\ell \in \mathbf{Z}$, they satisfy the polynomial equation $z^N - 1 = 0$. Since there are only N distinct roots of this equation in $\mathbf{C}$, in fact $z_{\ell+N} = z_\ell$ for all ℓ, and $z_0, z_1, z_2 = z_1^2 \ldots z_{N-1} = z_1^{N-1}$ are one complete set of distinct roots. (In abstract algebra terms, the set of Nth roots of unity is a cyclic subgroup of the multiplicative group of non-zero complex numbers, and z_1 is a generator.)

Discussion Questions

A) Show that the N-point DFT of any sequence is *periodic*, with period N: $F_{k+N} = F_k$ for all $k \in \mathbf{Z}$.

B) If we look at equation (1) and think of F_k as giving a way to approximate $\mathcal{F}[f]\left(\frac{2\pi k}{\Omega}\right)$, then notice from question A that essentially the maximum frequency value for which we can approximate $\mathcal{F}[f]$ with the N-point DFT is $\omega = \frac{2\pi N}{\Omega}$. Actually "even less is true." Namely, we really want to think of $\omega \in [\frac{-\pi N}{\Omega}, \frac{\pi N}{\Omega}]$. For signals like $\cos(\omega x)$, the DFT is symmetric, so that we are really talking about $0 \le \omega \le \frac{\pi N}{\Omega}$. Is this reasonable? Let's take $\Omega = 1$. Sketch the graph $y = \cos(12\pi x)$ on $[0, 1]$. Mark the points on the graph we would be getting by sampling with $N = 12$ and $N = 4$. Use what you see to answer the following questions:

   1) Find a different signal that "matches" the samples from $\cos(12\pi x)$ at $x = j/4$ (the sampling points with $N = 4$). Is there one of the form $\cos(\omega x)$ with $\omega < 12\pi$?
   2) Find a different signal that "matches" the samples from $\cos(12\pi x)$ at $x = j/12$ (the sampling points with $N = 12$). Is there one of the form $\cos(\omega x)$ with $\omega < 12\pi$?

The name for the phenomenon you see in parts 1 and 2 is *aliasing.* An *alias* of a signal is a different signal that has the same sampled values.

3) With $\Omega = 1$, and N given, why is $\omega = \frac{\pi N}{\Omega} = \pi N$ "special" here? And conversely, if $\Omega = 1$ is given and you have a signal $\cos(\omega x)$, how often should you sample (i.e. how big does N have to be) to make sure that there is no *lower-frequency* alias for your sampled values?

There is an extremely important generalization of this train of thought called the *Shannon-Nyquist Sampling Theorem.* The statement and proof will appear on this semester's Final Problem Set.

C) If f_j and g_j are sequences indexed by $j \in \{0, 1, \ldots, N-1\}$, then we define their *circular convolution* to be the sequence $h = f * g$ where

$$h_j = \sum_{i=0}^{N-1} f_i g_{j-i} \tag{2}$$

where, when $j - i$ falls outside the range $\{0, 1, \ldots, N-1\}$, we take the index *modulo* N. (Alternatively, we could think of making extended periodic sequences using f and g, then applying the convolution formula (2) to the periodic extensions.)

1) Compute the convolution of the 6-point sequences $\{f_j\} = 3, 2, 1, 1, 2, 3$ and $\{g_j\} = -1, 1, -1, 1, -1, 1$.

2) Show (in general) that if F_k, G_k, and H_k are the N-point DFT's of f_j, g_j, and $h_j = (f * g)_j$ respectively, then for all k,

$$H_k = 2\pi F_k G_k$$

(just like the convolution formula for the ordinary Fourier transform).

D) There is also a DFT version of the Fourier Inversion Formula. Here's how it works.

1) Show that $z_\ell = e^{\frac{2\pi i \ell}{N}}$ satisfies:

$$1 + z_\ell + z_\ell^2 + \cdots + z_\ell^{N-1} = \begin{cases} N & \text{if } \ell \equiv 0 \bmod N \\ 0 & \text{otherwise} \end{cases}$$

(Hint: If you have a polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + z^m,$$

with distinct roots $z = z_1, \ldots, z_m$ in $\mathbf{C}$, what is the sum of the roots?)

2) If F_k is the N-point DFT of f_j, show that

$$f_j = \frac{2\pi}{N} \sum_{k=0}^{N-1} F_k \, e^{\frac{-2\pi i (kj)}{N}}.$$

(The right hand side is the "DIFT" of the sequence F_k.)

Assignment

Group write-ups due Tuesday, April 18, if possible