Mathematics 174 – Applied Mathematics 2
Discussion 5 – Properties of the Fourier Transform
March 28, 2000

Background

We have introduced the Fourier Transform of a complex-valued $f(x)$ on $\mathbf{R}$:

$$\mathcal{F}[f](\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{i\omega x} \, dx.$$

If f is piecewise continuous and in $L^1(\mathbf{R})$ (that is, if $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$), then the Fourier transform of f is another well-defined function on $\mathbf{R}$.

For instance, we have seen that if

$$f(x) = \begin{cases} 1 & \text{if } x \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

then

$$\mathcal{F}[f](\omega) = \frac{\sin(a\omega)}{\pi\omega}.$$

The function $\frac{\sin(t)}{\pi t}$ is often called $\operatorname{sinc}(t)$, so the above formula for $\mathcal{F}[f](\omega)$ could also be written as

$$\mathcal{F}[f](\omega) = a \operatorname{sinc}(a\omega).$$

Also, if $f(x) = e^{-x^2}$ is the "standard" Gaussian, then

$$\mathcal{F}[f](\omega) = \frac{1}{\sqrt{4\pi}} e^{-\omega^2/4}.$$

In addition, we have seen the general Shift Property of the Fourier Transform: If $h(x) = f(x+a)$ is a horizontal shift of f, then

$$\mathcal{F}[h](\omega) = e^{-ia\omega} \mathcal{F}[f](\omega).$$

Comment: In communications and electrical engineering, the process of multiplying a function $g(t)$ by an exponential $e^{\pm iat}$ is called *modulation*. It corresponds to the idea of *amplitude modulation* (AM) for radio transmissions, where the signal $g(t)$ would be multiplied onto a "carrier wave" $\cos(at)$ (the real part of $e^{iat} = \cos(at) + i\sin(at)$) before it is transmitted as $g(t)\cos(at)$.

Discussion Questions

A) Consider the function $f(x) = e^{-|x|}$. Show that f is piecewise continuous and in $L^1(\mathbf{R})$, and compute its Fourier Transform.

B) Determine the following additional properties of the Fourier Transform and prove your statements.

1) What is the relationship between $\mathcal{F}[cf+g](\omega)$, $\mathcal{F}[f](\omega)$, and $\mathcal{F}[g](\omega)$?
2) What is the relationship between the Fourier Transform of $f(x)$ and the Fourier transform of a *modulation* of $f(x)$, $h(x) = e^{iax}f(x)$?
3) What is the relationship between the Fourier Transforms of $f(x)$ and a *horizontally scaled version* of $f(x)$, $h(x) = f(cx)$? Be careful about the sign of c!

C) If f is a real-valued function, then the real and imaginary parts of its Fourier Transform are

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\ dx, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\ dx$$

respectively. Give a sufficient condition for the Fourier Transform of f to be real (imaginary part zero). Similarly, give a sufficient condition for the Fourier Transform of f to be purely imaginary (real part zero).

Assignment

Group write-ups due next class, Thursday, March 30.