Mathematics 174 – Applied Mathematics 2
Discussion/Lab 4 – Applications of Eigenfunction Expansions
February 29, 2000

Background

Last Thursday, using eigenfunction expansions, we saw that it was possible to solve the following type of boundary value problem for the 1D heat equation with a heat source term:

$$\begin{cases} u_t = ku_{xx} + q(x,t) \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = g(x) \end{cases}$$

by expanding $u(x,t)$ and $q(x,t)$ in terms of the eigenfunctions $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ for the associated homogeneous problem (i.e. the same type of problem, but with $q(x,t)$ replaced by 0). If $u(x,t) = \sum_n a_n(t)\phi_n(x)$ and $q(x,t) = \sum_n c_n(t)\phi_n(x)$, then the coefficients $a_n(t)$ must satisfy the first order linear ODE's:

$$\frac{da_n(t)}{dt} + \lambda_n k a_n(t) = c_n(t).$$

Hence by the general method for solving equations of this type from Problem Set 1,

$$a_n(t) = e^{-\lambda_n kt}\left(a_n(0) + \int_0^t e^{\lambda_n kt} c_n(t)\ dt\right)$$

The values $a_n(0)$ are determined by the expansion of the initial temperature $g(x)$ in terms of the $\phi_n(x)$:

$$g(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x).$$

Today, we want to see that this method is actually very general: it applies in cases where the eigenfunctions are different from the sines above (and where there is more than one space dimension), and also to other equations (e.g. the wave equation).

Discussion/Lab Questions

A) First, let's consider a BVP for the 2D Heat Equation on a circular plate $0 \le r \le a$, with heat sources given by $q(r,\theta,t)$

$$\begin{cases} u_t = k\nabla^2 u + q(r,\theta,t) \\ u(a,\theta,t) = 0 \\ u(r,\theta,0) = \alpha(r,\theta) \end{cases} \tag{1}$$

From the associated homogenous problem ($q = 0$), we know the "2D eigenfunctions" (i.e. functions $\phi(r,\theta)$ which are solutions of $\nabla^2\phi = -\lambda\phi$, the equation obtained by *separation of t from the space variables*). Since there are two space dimensions here, we need two indices to list all of them:

$$\phi_{mn}(r,\theta) = \begin{cases} J_m\left(\frac{z_{mn}r}{a}\right)\cos(m\theta) \\ J_m\left(\frac{z_{mn}r}{a}\right)\sin(m\theta) \end{cases}$$

We have $\nabla^2\phi_{mn} = -\lambda_{mn}\phi_{mn}$, where $\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$. Note: this way of writing things is not really satisfactory, since we are not distinguishing between the "sin" and "cos" forms of ϕ in the notation. *However, the exact form of $\phi_{mn}(r,\theta)$ will not be all that relevant in what we have to do here, so let's not worry about that!*

The Question: Suppose we know the expansions

$$q(r,\theta,t) = \sum_{m,n} c_{mn}(t)\phi_{mn}(r,\theta)$$

and

$$\alpha(r,\theta) = \sum_{m,n} b_{mn}\phi_{mn}(r,\theta)$$

(the b_{mn} are constants, since α does not depend on t – it's the initial temperature distribution). Determine $a_{mn}(t)$ so that

$$u(r,\theta,t) = \sum_{m,n} a_{mn}(t)\phi_{mn}(r,\theta)$$

solves the problem (1). (Assume all derivatives can be interchanged with summations as necessary.)

B) Next we want to consider an inhomogeneous 1D wave equation:

$$u_{tt} = c^2u_{xx} + q(x,t).$$

Solutions of this type of equation represent the vibrations of strings, in the presence of an *external driving force* $q(x,t)$ (i.e. a force different from the tension forces exerted by the string itself). Examples would be forces exerted by motions of the air around the string, forces exerted by another object such as a bow in contact with the string, etc. In particular, suppose we have a string fixed at its endpoints, with given initial position and velocity, and a known driving force q. The displacement $u(x,t)$ satisfies

$$(2) \qquad \begin{cases} u_{tt} = c^2u_{xx} + q(x,t) \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

We write the eigenfunctions of the associated homogeneous equation as $\phi_n(x)$ as before, and write λ_n for the associated eigenvalue.

1) Following what we did with the heat equation, suppose that

$$f(x) = \sum_n b_n \phi_n(x), \qquad g(x) = \sum_n c_n \phi_n(x)$$

(here, b_n and c_n are constants for each n) and

$$q(x,t) = \sum_n d_n(t)\phi_n(x).$$

What differential equations must the coefficients $a_n(t)$ in the expansion $u(x,t) = \sum_n a_n(t)\phi_n(x)$ satisfy if u is to be a solution of (2)?

From now on, let's suppose that $q(x,t)$ is of the form $q(x,t) = \cos(\omega t)$ – a periodic driving force with frequency ω. A solution of the form $a_n(t)\phi_n(x)$ (no summation!) can be viewed as one of the fundamental modes of vibration for the string in this situation. Our next goal is to understand the form of these solutions, and how they depend on the driving frequency ω.

2) From part 1, we know that each a_n satisfies a 2nd order ODE (the equation should depend on n too, since the corresponding eigenvalue λ_n will appear in it). We can use a numerical method to generate approximate solutions of these equations! Plot $a_n(t)$ versus t with Maple, following what we did with the `DEplot` command in Lab/Problem Set 1. Take $c = 1$, $L = 3$ for all plotting, and make the initial conditions $a_n(0) = 1, a_n'(0) = 0$ (these are chosen abitrarily; in an actual problem, these values would be determined from the Fourier coefficients of the functions $f(x), g(x)$ in (2)).

$$\begin{cases} n = 2 & \omega = 1, 1.5, 2, 2.094, 3, 4 \\ n = 3 & \omega = 1, 2, 3.1, 3.1416, 4, 5 \\ n = 4 & \text{your choice – try to find value(s) of } \omega \\ & \text{where something similar to the behavior at} \\ & \omega = 2.094 \text{ for } n = 2\text{, and } \omega = 3.1416 \\ & \text{for } n = 3 \text{ occurs.} \end{cases}$$

Maple technical hint: To minimize the work you need to do, you can set up a *function* that takes ω as its argument, and produces the plot you want as the output.

3) Discuss your results. The usual name for the special behavior at $\omega \doteq 2.094$ for $n = 2$, and $\omega \doteq 3.1416$ for $n = 3$ is *resonance*. Why is that an apt name? (What would be happening to a string vibrating this way??)

4) The solutions of the ODE in part 2 can also be expressed analytically through a method called *variation of parameters*. The idea is this. Just as in the 1st order linear case that we saw in Problem Set 1, the general solution of

$$\frac{d^2 a_n}{dt^2} + c^2 \lambda_n a_n = \cos(\omega t) \tag{3}$$

can be written as the *sum of* the general solution of the associated homogeneous equation

$$\frac{d^2 a_n}{dt^2} + c^2 \lambda_n a_n = 0,$$

namely

$$\alpha_1 \cos(c\sqrt{\lambda_n} t) + \alpha_2 \sin(c\sqrt{\lambda_n} t)$$

(α_1, α_2 arbitrary constants), and any particular solution of (3). To find a particular solution of (3), we can use the fact that if $a(t) = A\cos(\omega t)$, then the second derivative of $a(t)$ is another constant times the same cosine. What should A be to yield a solution of (3)? (Note: "varying the parameter A" yields the solution – hence the name of the method.)

5) Use this method to solve (3) with $n = 2$ and $\omega = 1$:

$$\frac{d^2 a_2}{dt^2} + c^2 \left(\frac{2\pi}{3}\right)^2 a_n = \cos(t)$$

together with the initial conditions $a_2(0) = 1$, $a_2'(0) = 0$. Plot your solution using the formula you get and compare with the corresponding numerical approximation you saw in part 2.

6) In finding the formula for A in part 4, you should have had to *divide* at some point. That step is not valid if the quantity you divided by is zero!! For which ω does that happen? These are called the *resonant frequencies of the string*. There is a corresponding sequence of resonant frequencies for any physical object, determined by the eigenvalues of the Laplacian on it(!)

7) If ω is one of the resonant frequencies, there is a particular solution of (3) that has a different form. Show that in this case,

$$\frac{1}{2\omega} t \sin(\omega t)$$

is a solution of (3). Again solve (3) with $n = 2$:

$$\frac{d^2 a_2}{dt^2} + c^2 \left(\frac{2\pi}{3}\right)^2 a_n = \cos(\omega t)$$

together with the initial conditions $a_2(0) = 1$, $a_2'(0) = 0$, in the case that $\omega = c\frac{2\pi}{3}$. Plot your solution (taking $c = 1$) and compare with the corresponding numerical approximation you saw in part 2.

Assignment

Group write-ups due Thursday, March 16.