Mathematics 174 – Applied Mathematics 2
Discussion 3/Problem Set 2 – Bessel Functions
February 10, 2000

Background

Recall that last time we began studying the 2D wave equation for a circular membrane with its outer edge (at $r = a$) fixed at zero vertical displacement. By separation of variables, we were led to the Sturm-Liouville form equation:

$$\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{m^2}{r}f = -\lambda r f \tag{1}$$

with the singular boundary conditions $|f(0)| < \infty$ and $f(a) = 0$. Because $p = r$ and $\sigma = r$ are not strictly positive on $[0, a]$, and since $q = -\frac{m^2}{r}$ has a discontinuity at $r = 0$, this *is not a regular Sturm-Liouville problem.* However, we will see that the statements in the Sturm-Liouville theorem *are still true here.*

After multiplying by r and making the change of variable $z = \sqrt{\lambda}r$, equation (1) reduced to *Bessel's equation of order* m:

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2)f = 0.$$

We will be most interested in the case that m is a positive integer. However, Bessel's equation makes sense as a differential equation for all $m \in \mathbf{R}$. We derived a formal series solution of this equation:

$$J_m(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!(m+\ell)!}\left(\frac{z}{2}\right)^{2\ell+m} \tag{2}$$

called the *Bessel function of the first kind, of order* m.

Discussion Questions

A) Our derivation of the series expansion (2) for $J_m(z)$ was completely "formal" – it was based only on the recurrence relation for the coefficients. In order to say that the infinite series in (2) determines a well-defined function of z on some interval containing zero, *what else do we have to show*? On what interval $(-L, L)$ does (2) specify a well-defined function of z?

B) The part of the Sturm-Liouville theorem that is easiest to prove (even without regularity) is the *orthogonality relation for eigenfunctions*. Assume $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of (1) – that is, assume there exist non-zero f_1 and f_2 satisfying

$$\begin{cases} L(f_1) = \frac{d}{dr}\left(r\frac{df_1}{dr}\right) - \frac{m^2}{r}f_1 = -\lambda_1 r f_1 \\ L(f_2) = \frac{d}{dr}\left(r\frac{df_2}{dr}\right) - \frac{m^2}{r}f_2 = -\lambda_2 r f_2 \end{cases}$$

and $f_1(a) = f_2(a) = 0$ (and $f_1(0), f_2(0)$ finite). Deduce that

$$\langle f_1, f_2 \rangle_r = \int_0^a f_1(r) f_2(r) r \; dr = 0.$$

Problem Set 2

C) Write out the first 6 non-zero terms in the expansions of $J_0(z)$ and $J_1(z)$. Plot the corresponding functions together on the interval $[0, 4]$.

D) It is a general fact that for a function defined by a convergent power series (as in (2)), the derivative can be found by term-by-term differentiation. Show that

$$\frac{d}{dz}(z^{-m} J_m(z)) = -z^{-m} J_{m+1}(z)$$

for all integers $m \geq 0$ and similarly,

$$\frac{d}{dz}(z^m J_m(z)) = z^m J_{m-1}(z)$$

for all integers $m \geq 1$. If $m = 0$ in particular, this says $J_0'(z) = -J_1(z)$. Does this agree with your plots in part C?

E) Prove the following recurrence relation for the Bessel functions

$$J_{n+1}(z) = \frac{2n}{z} J_n(z) - J_{n-1}(z)$$

(Hint: Combine the two relations from question D.)

So far we have concentrated on integer values m. The series expansion for $J_m(z)$ given above can also be extended to non-integer m if we replace the factor $(m+\ell)!$ by a value of an important special function called the *gamma function*. The gamma function is usually defined as follows: For $x > 0$,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \; dt$$

It *interpolates values of the factorial* in the sense given in part 1 of the following problem.

F)

1) Show that $\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$ (Hint: Think parts!) Deduce that for all integers $n \geq 0$, $\Gamma(n+1) = n!$.

2) Explain how the relation $\Gamma(x+1) = x\Gamma(x)$ can also be used to define $\Gamma(x)$ for $x < 0$. Note though: $\Gamma(0)$ is undefined. This implies that $\Gamma(n)$ is not defined either if $n < 0$ is an integer. Also: the gamma function is built-in in Maple. It is called `GAMMA` (all capitals). Be careful if you try to plot it on an interval containing a negative integer though. The gamma function has vertical asymptotes at those points.

Now, we can define

$$J_m(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\Gamma(m+\ell+1)} \left(\frac{z}{2}\right)^{2\ell+m} \tag{3}$$

and the series in (3) makes sense for non-integer m as well.

G) In this question we will see that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{4}$$

1) First by definition,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2} dt$$

Let $t = u^2$, and change variables in the integral.

2) Next, take two "copies" of your integral in part 1, written with different variables of integration u, v. If you multiply the copies together, the result can be written as a double integral over the first quadrant of the u, v plane. It can be evaluated by converting to polar coordinates (the polar limits are $r \in [0, \infty)$, $\theta \in [0, \pi/2]$). What value do you get? Deduce (4) above.

H) By calculations similar to those in question G it can be shown that for any integer $n \geq 1$,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \sqrt{\pi}.$$

This implies that if m is of the form (odd integer)/2, then the Bessel function $J_m(z)$ can be expressed in terms of ordinary trigonometric functions. For instance, show that

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z)$$

and

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z)$$

Find a similar formula for $J_{3/2}(z)$.

Assignment

Group write-ups for questions A,B, and individual write-ups for the others, due Thursday, February 17.