Mathematics 174 – Applied Mathematics 2
Discussion 2 – The Rayleigh Quotient and Applications
February 3, 2000

Background

We consider a general regular Sturm-Liouville boundary value problem

$$(1) \qquad \begin{cases} L(\phi) = \frac{d}{dx}\left(p(x)\phi\right) + q(x)\phi = -\lambda\sigma(x)\phi \\ \beta_1\phi(a) + \beta_2\phi'(a) = 0 \\ \beta_3\phi(b) + \beta_4\phi'(b) = 0 \end{cases}$$

Let V denote the vector space of C^2 functions on $[a, b]$ satisfying the boundary conditions in (1). For any $u \in V$ (any *test function*) we can define the *Rayleigh Quotient*:

$$RQ(u) = \frac{-\langle L(u), u\rangle_1}{\langle u, u\rangle_\sigma}$$

At the end of class on Tuesday, we stated

Theorem. *Let λ_1 be the minimal eigenvalue of (1). Then*

$$\lambda_1 = \min_{u \in V} RQ(u)$$

Today we want to prove this result and use it to study several boundary value problems for the 1-dimensional heat equation.

Discussion Questions

A) To deduce the theorem, proceed as follows:

1. First, if $u \in C^2([a, b])$ then the last portion of the Sturm-Liouville theorem implies that the eigenfunction expansion

$$u(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x)$$

(a_n are the generalized Fourier coefficients of u) converges to $u(x)$ for all $x \in (a, b)$. Substitute this expression for u into the Rayleigh quotient and work with the result to show that

$$RQ(u) = \frac{\sum_{n=1}^{\infty} a_n^2\lambda_n\langle\phi_n, \phi_n\rangle_\sigma}{\sum_{n=1}^{\infty} a_n^2\langle\phi_n, \phi_n\rangle_\sigma}$$

2. Recall that $\lambda_1 < \lambda_n$ for all $n > 1$. Use this to show that $RQ(u) \geq \lambda_1$ for all u.
3. For which u does $RQ(u)$ achieve its minimum value of λ_1?

B) Use the Rayleigh Quotient to show that the minimal eigenvalue λ_1 of

$$\begin{cases} \frac{d^2\phi}{dx^2} = -\lambda(x^4 + x^2 + 1)\phi \\ \phi(0) = 0 \\ \phi'(1) = 0 \end{cases}$$

is strictly positive. (You will need to rule out $\lambda_1 = 0$ with a separate argument.)

C) In our study of boundary value problems for the 1-D heat equation last semester, we concentrated almost exclusively on the cases where the material of the rod had constant thermal properties (that is c, ρ, K_0 were assumed constant). One nice aspect of the Sturm-Liouville theory is that it lets us remove that restriction. Namely, let's now consider a rod extending along the interval $[0, L]$ in x, with no heat sources. Then the temperature function $u(x, t)$ satisfies

$$c(x)\rho(x)u_t = (K_0(x)u_x)_x \tag{2}$$

1. In all cases of physical interest, the functions $c(x), \rho(x), K_0(x)$ have *strictly positive* values on $[0, L]$. Show that separation of variables $u(x, t) = h(t)\phi(x)$ leads to a regular Sturm-Liouville problem for $\phi(x)$ in the case that the ends of the rod are maintained at zero temperature for all times, and in the case that the ends of the rod are insulated. (What functions play the roles of the p, q, σ in the general form (1) here?)
2. In the case that the ends of the rod are insulated, show that the minimal eigenvalue is $\lambda_1 = 0$. (Hint: the Rayleigh Quotient can be used to rule out the possibility $\lambda_1 < 0$.)
3. Still in the case that the ends of the rod are insulated, let $u(x, 0) = f(x)$ be the initial temperature distribution. By writing the solution of (2) and the boundary conditions $u_x(0, t) = 0, u_x(L, t) = 0$ using the Sturm-Liouville eigenfunctions and taking a limit as $t \to \infty$, determine the corresponding equilibrium temperature. (Note: you don't have specific $\phi_n(x)$ to work with here because $c(x), \rho(x), K_0(x)$ are not given explicitly. Your answer should be completely general.)
4. Now assume $u(0, t) = u(L, t) = 0$ for all t. Using the Rayleigh Quotient, show that if
$$\begin{cases} 0 < m \le K_0(x) \le M & \text{for all } x \in [0, L] \text{ and} \\ 0 < r \le c(x)\rho(x) \le R & \text{for all } x \in [0, L] \text{ and} \end{cases}$$
then
$$\frac{m}{R} \cdot \left(\frac{\pi}{L}\right)^2 \le \lambda_1 \le \frac{M}{r} \cdot \left(\frac{\pi}{L}\right)^2$$
In particular, since $m, R > 0$, we have $\lambda_1 > 0$.
5. Still assuming $u(0, t) = u(L, t) = 0$, suppose $u(x, 0) = f(x)$. The solution of the boundary value problem will look like
$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\phi_n(x)$$

where

$$a_n = \frac{\langle f, \phi_n \rangle_\sigma}{\langle \phi_n, \phi_n \rangle_\sigma}$$

for the appropriate weight function σ. Which term here decays to zero *most slowly*? (As $t \to \infty$, this will be the "most visible" term in the solution.)

Assignment

Group write-ups due Thursday, February 10.