Mathematics 174 – Applied Mathematics 2
Discussion 1 – Sturm-Liouville Eigenvalue Problems
January 20, 2000

Background

Last time, we introduced the standard Sturm-Liouville boundary value problem:

$$\begin{cases} \frac{d}{dx}\left(p(x)\frac{d\phi(x)}{dx}\right) + q(x)\phi(x) + \lambda\sigma(x)\phi(x) = 0 \\ \beta_1\phi(a) + \beta_2\phi'(a) = 0 \\ \beta_3\phi(b) + \beta_4\phi'(b) = 0 \end{cases}$$

We say the problem is *regular* if p, q, σ are continuous functions on $[a, b]$, and $p(x), \sigma(x) > 0$ for all $x \in [a, b]$. For a regular Sturm-Liouville problem, we stated the following results:

1. The set of eigenvalues (the set of all constants λ for which a non-zero solution $\phi(x)$ of the problem exists) forms a *sequence* of real numbers. This sequence has a smallest element λ_1, and can be listed in increasing order: $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. When this is done, $\lambda_n/n \to \infty$ as $n \to \infty$ (so in particular, λ_n grows faster than n as $n \to \infty$).
2. For each n, the eigenspace for λ_n is 1-dimensional. Call a basis eigenfunction $\phi_n(x)$. $\phi_n(x)$ has exactly $n-1$ distinct zeroes in the open interval (a, b) (and possibly additional zeroes at the endpoints).
3. The following orthogonality relation holds: If $m \neq n$
$$\int_a^b \phi_m(x)\phi_n(x)\sigma(x)\ dx = 0.$$
4. The system of eigenfunctions $\{\phi_1(x), \phi_2(x), \ldots\}$ has the *completeness* property. In particular, if f is a continuous function which satisfies $\int_a^b f(x)\phi_n(x)\sigma(x)\ dx = 0$ for all n, then $f(x)$ is identically zero on $[a, b]$.
5. If $f(x)$ is piece-wise smooth on $[a, b]$, then we can expand $f(x)$ using the eigenfunctions as $f(x) \sim \sum_{n=1}^{\infty} a_n\phi_n(x)$ where the "Fourier coefficients" a_n are given by:
$$a_n = \frac{\int_a^b f(x)\phi_n(x)\sigma(x)\ dx}{\int_a^b (\phi_n(x))^2\sigma(x)\ dx}.$$
Morover, as in the Pointwise Convergence Theorem for Fourier Series, this series converges to $\frac{1}{2}(f(x+) + f(x-))$ for all $x \in (a, b)$.

Today, we want to look at further examples, verify statements 1 - 3 (the rest are more difficult), and consider some ramifications of these ideas.

Discussion Questions

A) Consider the Sturm-Liouville problem

$$\begin{cases} \frac{d^2\phi(x)}{dx^2} + \lambda\phi(x) = 0 \\ \phi'(0) = 0 \\ \phi(L) = 0 \end{cases}$$

Identify the functions $p(x), q(x), \sigma(x)$ here. What are the eigenvalues and eigenfunctions for this problem? Show that all parts of statements 1,2,3 above are satisfied for the eigenvalues and eigenfunctions.

B) The second order ODE in the standard Sturm-Liouville problem can be rewritten as

$$\frac{d}{dx}\left(p(x)\frac{d\phi(x)}{dx}\right) + q(x)\phi(x) = -\lambda\sigma(x)\phi(x) \tag{1}$$

Write L for the *operator* on functions on the left side. That is,

$$L(\phi) = \frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi.$$

Show that L is a *linear operator* on functions.

C) Consider the 2nd order linear differential operator

$$\overline{L}(\phi) = x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx}$$

Take the equation $\overline{L}(\phi) = -\lambda\phi$ and divide both sides by x. The result can be rearranged to Sturm-Liouville form:

$$\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + \frac{1}{x}\phi = -\frac{\lambda}{x}\phi.$$

1) Suppose we add the boundary conditions $\phi(1) = 0$, $\phi(2) = 0$. Is this a *regular* Sturm-Liouville problem? Why or why not?
2) Verify that $\lambda_n = \left(\frac{n\pi}{\ln 2}\right)^2$ is an eigenvalue of this problem, with corresponding eigenfunction $\phi_n(x) = \sin(\sqrt{\lambda_n}\ln(x))$.
3) Verify that property 3 above (the orthogonality relation) is satisfied for the appropriate $\sigma(x)$.

D) The Sturm-Liouville operator L in question B looks rather special; however we will see that in fact it is quite general. An apparently more general second-order linear differential operator $\overline{L}$ on functions would look like this:

$$\overline{L}(\phi) = \frac{d^2\phi}{dx} + B(x)\frac{d\phi}{dx} + C(x)\phi$$

The equation $\overline{L}(\phi) = -\lambda\phi$ is the eigenvalue/eigenvector equation for the operator $\overline{L}$. In general, show that if we multiply both sides of the equation

$$\frac{d^2\phi}{dx^2} + B(x)\frac{d\phi}{dx} + C(x)\phi = -\lambda\phi$$

by the function $e^{\int B(x)\ dx}$ (called an "integrating factor"), then the result can be rewritten in Sturm-Liouville form for suitable functions $p(x), q(x), \sigma(x)$ (what are they)?

Assignment

Group write-ups, due in class on Thursday, January 27.