MATH 351 – Modern Algebra 1
Selected Solutions for Problem Set 7
November 3, 2018

4.46. (1) We are given that N and K are subgroups of index 2 of G with $N \neq K$. By Theorem 4.1 in Lee, note that both N an K are normal subgroups of G. By the result of Exercise 4.5, $N \cap K$ is also a normal subgroup of G. This follows because if $g \in G$ is arbitrary, and $a \in N \cap K$, then $a \in N$, so $gag^{-1} \in N$ and similarly $a \in K$, so $gag^{-1} \in K$. Hence $gag^{-1} \in N \cap K$, so $N \cap K$ is normal in G. The same argument just given also works if we only take $g \in N$. Hence $N \cap K$ is also a normal subgroup of N. Now by the *Second Isomorphism Theorem*, we have

$$N/(N \cap K) \cong NK/K.$$

We claim that $NK = G$. This is true because K has index 2 in G. In the set

$$NK = \{nk \mid n \in N, k \in K\},$$

if $n \in K$ as well, then $nK = K$, so we get $K \subset NK$. On the other hand, we are assuming $N \neq K$, so there are $n \in N$ that are in $G \backslash K$ as well. If $n \notin K$, then the coset $nK = G \backslash K$, the other left coset of K in G. So $G \backslash K \subset NK$ as well. Hence $K \cup (G \backslash K) = G \subseteq NK \subseteq G$. It follows that $NK = G$. Now we have proved what was required since

$$[N : N \cap K] = |N/(N \cap K)| = |NK/K| = |G/K| = 2.$$

(2) Now consider the mapping

$$\alpha : G \to G/N \times G/K$$
$$g \mapsto (gN, gK)$$

G/N and G/K have order 2 so each is isomorphic to $\mathbf{Z}_2$ by Corollary 4.2. We want to show that α is a surjective group homomorphism and identify the kernel as $N \cap K$ to show that $G/(N \cap K) \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. Here are the individual steps:

- We claim first that α is a group homomorphism. This is true because

  $$\alpha(g \cdot h) = ((g \cdot h)N, (g \cdot h)K) = (gN \cdot hN, gK \cdot hK) = (gN, gK) \cdot (hN, hK) = \alpha(g) \cdot \alpha(h),$$

  using the definition of the coset product in G/N and G/K.
- The kernel of α consists of the $g \in G$ such that $(gN, gK) = (N, K)$, the identity element of $G/N \times G/K$. But $(gN, gK) = (N, K)$ if and only if $g \in N$ and $g \in K$, so $g \in N \cap K$. Hence $\ker(\alpha) = N \cap K$.
- Next, we claim that $\alpha(G) = (G/N, G/K)$ (that is, that α is surjective). This follows by thinking about what part (1) of the problem says. We have $[N : N \cap K] = 2$. The same argument, but reversing the roles of N, K shows that $[K : N \cap K] = 2$ as well.

Moreover $[G : N] = [G : K] = 2$ by assumption. It follows that G decomposes into four disjoint subsets:

$$N \cap K, (G \setminus N) \cap K, N \cap (G \setminus K), \text{ and } (G \setminus N) \cap (G \setminus K).$$

All of these are nonempty since $[N : N \cap K] = [K : N \cap K] = 2$. The g in the first map to (N, K) under α, and the g in the other three subsets map to the other three elements in $G/N \times G/K$. Therefore α is surjective.

Comment: We actually saw an example of the pattern described here when we considered $G = D_8$, $N = \langle R_{90} \rangle$ and $K = \{R_0, R_{180}, F_1, F_2\}$. These are two unequal subgroups of order 4 in a group of order 8, so we're exactly in the situation of the problem. In this case, $N \cap K = \{R_0, R_{180}\} = Z(D_8)$ and we showed directly that

$$D_8/Z(D_8) \cong \mathbf{Z}_2 \times \mathbf{Z}_2.$$