MATH 351 – Modern Algebra 1
Selected Solutions – Problem Set 8
November 10, 2018

5.10. We are given that $N_1, \ldots, N_k$ are finite normal subgroups of some group G such that $\gcd(|N_i|, |N_j|) = 1$ whenever $i \neq j$ and we want to show that

$$N_1 \times \cdots \times N_k \cong N_1 N_2 \cdots N_k.$$

It is true that this is intimately connected with Lemmas 5.1 and 5.2 and Theorem 5.2. However there are some differences, too, that we must take into account. First, note that we are not assuming $G = N_1 \cdots N_k$, so it might not be true that G is the internal direct product of the subgroups N_i. The product $N_1 \cdots N_k$ could be some proper subgroup of G. Second, the hypothesis $\gcd(|N_i|, |N_j|) = 1$ here is different from the condition that appears in the definition of the internal direct product. Hence there are several different ways to proceed here.

Solution 1: (The "slickest" way.) We argue by induction on k. If $k = 1$, there is nothing to show. If this seems unsatisfying as a base case, consider the case $k = 2$. Then note that $\gcd(|N_1|, |N_2|) = 1$ implies that $N_1 \cap N_2 = \{e\}$ by Lagrange's theorem, since $N_1 \cap N_2$ is a subgroup of both N_1 and N_2. Then $N_1 \times N_2 \cong N_1 N_2 = H$ follows from Theorem 5.2 applied to the group $H = N_1 N_2$ (not G), since H is the internal direct product of its subgroups N_1 and N_2. Now assume the statement holds for all collections of k subgroups satisfying the given conditions, and consider $N_1, \ldots, N_k, N_{k+1}$. By the induction hypothesis, $N_1 N_2 \cdots N_k$ is isomorphic to $N_1 \times N_2 \times \cdots \times N_k$. In particular,

$$|N_1 N_2 \cdots N_k| = |N_1||N_2| \cdots |N_k|.$$

It follows that $\gcd(|N_1 N_2 \cdots N_k|, |N_{k+1}|) = 1$, since $\gcd(|N_i|, |N_{k+1}|) = 1$ for all $1 \leq i \leq k$. But this implies

$$(N_1 N_2 \cdots N_k) \cap N_{k+1} = \{e\},$$

as above. Since $N_1 N_2 \cdots N_k$ is normal in G, the proof that

$$(N_1 \times N_2 \times \cdots \times N_k) \times N_{k+1} \cong (N_1 N_2 \cdots N_k) N_{k+1}$$

will follow exactly as in the case $k = 2$ done above.

Solution 2: (This one is essentially repeating the proof of Theorem 5.2, which is OK, but less "elegant.") Consider the mapping

$$\begin{aligned} \alpha : N_1 \times N_2 \times \cdots \times N_k &\to N_1 \cdots N_k \\ (n_1, n_2, \ldots, n_k) &\mapsto n_1 n_2 \cdots n_k \end{aligned}$$

Then

- α is surjective by definition.

- α is a group homomorphism by Lemma 5.1 and the same argument as in the text in the proof of Theorem 5.2:

$$\begin{aligned}\alpha(n_1n_1', n_2n_2', \ldots, n_kn_k') &= (n_1n_1')(n_2n_2')\cdots(n_kn_k') \\ &= (n_1n_2\cdots n_k)(n_1'n_2'\cdots n_k') \\ &= \alpha(n_1, n_2, \ldots, n_k)\alpha(n_1', n_2', \ldots, n_k'),\end{aligned}$$

since Lemma 5.1 implies that the n_i' and the n_j all commute with each other.
- Finally, we show that α is injective, using induction on k. If $k = 1$, there is nothing to prove. So assume that the result is true whenever $k = \ell$ and consider products of $\ell + 1$ subgroups. To show α is injective, it suffices to show that $\ker(\alpha) = (e, e, \ldots, e)$. So consider $(n_1, n_2, \ldots, n_{\ell+1})$ such that

$$\alpha(n_1, n_2, \ldots, n_\ell, n_{\ell+1}) = n_1n_2\cdots n_\ell n_{\ell+1} = e.$$

This implies

$$n_{\ell+1}^{-1} = n_1n_2\cdots n_\ell \in (N_1N_2\cdots N_\ell) \cap N_{\ell+1}.$$

In the "slick" solution, we saw that $(N_1N_2\cdots N_\ell) \cap N_{\ell+1} = \{e\}$ because $N_1N_2\cdots N_\ell$ is isomorphic to $N_1 \times N_2 \times \cdots \times N_\ell$ so the order of $N_1N_2\cdots N_\ell$ is the product of the orders of the N_i, hence is relatively prime to $|N_{\ell+1}|$. The same is true here. Hence that implies $n_{\ell+1} = e$ and then by induction $n_1n_2\cdots n_\ell = e$ implies $n_1 = n_2 = \cdots = n_\ell = e$ as well.

5.18. In the cyclic group $G = \langle a \rangle$ of order $90 = 2 \cdot 3^2 \cdot 5$, the subgroup of 2-elements is $H_2 = \{e, a^{45}\} = \langle a^{45} \rangle$. The subgroup of 3-elements is $\langle a^{10} \rangle$. (Note that a^{10}, a^{20}, a^{40}, a^{50}, a^{70}, a^{80} have order 9, while a^{30} and a^{60} have order 3, and e has order 1. So all the elements of $\langle a^{10} \rangle$ are 3-elements and they the only ones.) Similarly, the subgroup of 5-elements is $\langle a^{18} \rangle$. Since G is the internal direct product of its subgroups H_2, H_3, H_5 by Lemma 5.5, we know there is exactly one way to write $a = bcd$ with $b \in H_2$, $c \in H_3$ and $d \in H_5$. A little experimentation gives

$$b = a^{45}, \quad c = a^{10}, \quad d = a^{36}.$$

(Note, it would also be possible to solve this by setting up a congruence on the exponents writing $b = a^{45i}, c = a^{10j}, d = a^{18k}$. We want to find i, j, k with $45i + 10j + 18k \equiv 1 \bmod 90$, and $i = 1, j = 1, k = 2$ is clearly one solution.)