MATH 351 – Modern Algebra I
Selected Solutions for Problem Set 10
December 7, 2018

7.18. Let G be an infinite group and let H be the subset of G consisting of the elements with only finitely many conjugates. Equivalently H is the subset of $a \in G$ such that $|C_a|$ is finite. We have to show that H is a subgroup of G.

- First, we consider e. For all $g \in G$, $geg^{-1} = e$. Therefore $C_e = \{e\}$ and $|C_e| = 1$. Since this is finite, $e \in H$.
- Now let $a, b \in H$ and consider C_{ab}. For all $g \in G$, we have using associativity

$$gabg^{-1} = ga(g^{-1}g)bg = (gag^{-1})(gbg^{-1}).$$

  The factor on the left is an element of C_a and the factor on the right is an element of C_b. Hence when we form these products, we obtain *at most* $|C_a| \cdot |C_b|$ different elements of G. There could be also strictly fewer than $|C_a| \cdot |C_b|$ in some cases, but in any case, the number $|C_{ab}|$ is finite. This shows $ab \in H$.
- Now let $a \in H$ and consider $|C_{a^{-1}}|$. The elements of this class all have the form $ga^{-1}g^{-1}$ for $g \in G$. Note that the inverse of this element of G is

$$(ga^{-1}g^{-1})^{-1} = gag^{-1},$$

  which is one of the elements of $|C_a|$. Since we began by assuming $a \in H$, there are only finitely many different values obtained for gag^{-1}. That implies there are only finitely many conjugates of a^{-1} too since each element of G has a unique inverse in G. That shows $a^{-1} \in H$.

Putting together these three points, we get that H is a subgroup of G.

7.24. Most people saw correctly that you want to use Sylow II to say that any two Sylow 2-subgroups H_1 and H_2 must be conjugate. But then we must say why (or prove directly that) if $H_2 = gH_1g^{-1}$ for two subgroups H_1, H_2 of G and some element $g \in G$, then H_1 and H_2 must be *isomorphic* as groups. Here's how to show that. Consider the conjugation mapping α:

$$\alpha : H_1 \to H_2$$
$$h \mapsto ghg^{-1}$$

We claim that α is an isomorphism. First, it is a group homomorphism because if $h, k \in H_1$, then

$$\alpha(hk) = ghkg^{-1} = gh(g^{-1}g)hg^{-1} = (ghg^{-1})(gkg^{-1}) = \alpha(h)\alpha(k).$$

The mapping α is surjective with image H_2 by Sylow II. Moreover, since $|H_1| = p^n = |H_2|$ if $|G| = p^n r$ with $\gcd(p, r) = 1$, once we know α is surjective, then α must be injective as well. Hence α is an isomorphism of groups.

But this shows that it is impossible for one Sylow 2-subgroup $H_1 \cong \mathbf{Z}_4$ and another Sylow 2-subgroup $H_2 \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. The groups $\mathbf{Z}_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ are not isomorphic, by the fundamental theorem of finite abelian groups.

7.30. Let $|G| = 56 = 2^3 \cdot 7$. We claim that G must have a normal subgroup other than $\{e\}$ and G, so G is not simple. Let $p = 7$. By Sylow III, the number of Sylow 7-subgroups is $\equiv 1 \bmod 7$ and that number divides 8. Thus both 1 and 8 Sylow 7-subgroups are possible. If there is just one, then as usual, Sylow II implies that subgroup is normal and we are done. On the other hand, if there are 8 Sylow 7-subgroups in G, each of order $= 7$, then since 7 is prime, each of the Sylow 7-subgroups is cyclic and there are 6 elements of order 7 in each of them. Distinct Sylow 7-subgroups can only intersect in the identity element. Counting up the total number of elements in the union of those subgroups we get $1 + 8 \cdot 6 = 49$. This leaves only 7 other elements of G, and those 7 elements together with e make up exactly one subgroup of order 8 by Sylow I (there exists at least one subgroup with $2^3 = 8$ elements, but there is no room for more than one in this case(!)). Hence if there are 8 Sylow 7-subgroups, there must be only one Sylow 2-subgroup, hence that Sylow 2-subgroup is normal by Sylow II. Hence G is not simple in this case either.